The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2017 Euclid Contest

Thursday, April 6, 2017
(in North America and South America)

Friday, April 7, 2017
(outside of North America and South America)

Solutions

1. (a) Since $5(2) + 3(3) = 19$, then the pair of positive integers that satisfies $5a + 3b = 19$ is $(a, b) = (2, 3)$.

(b) We list the first several powers of 2 in increasing order:

n	1	2	3	4	5	6	7	8	9	10	11
2^n	2	4	8	16	32	64	128	256	512	1024	2048

Each power of 2 can be found by multiplying the previous power by 2.
From the table, the smallest power of 2 greater than 5 is $2^3 = 8$ and the largest power of 2 less than 2017 is $2^{10} = 1024$. Since 2^n increases as n increases, there can be no further powers in this range.
Therefore, the values of n for which $5 < 2^n < 2017$ are $n = 3, 4, 5, 6, 7, 8, 9, 10$.
There are 8 such values of n.

(c) Each of the 600 Euros that Jimmy bought cost \$1.50.
Thus, buying 600 Euros cost $600 \times \$1.50 = \$900$.
When Jimmy converted 600 Euros back into dollars, the rate was $\$1.00 = 0.75$ Euro.
Therefore, Jimmy received 600 Euros $\times \dfrac{\$1.00}{0.75 \text{ Euros}} = \dfrac{\$600}{0.75} = \$800$.
Thus, Jimmy had $\$900 - \$800 = \$100$ less than he had before these two transactions.

2. (a) Since $x \neq 0$ and $x \neq 1$, we can multiply both sides of the given equation by $x(x-1)$ to obtain $\dfrac{5x(x-1)}{x(x-1)} = \dfrac{x(x-1)}{x} + \dfrac{x(x-1)}{x-1}$ or $5 = (x-1) + x$.
Thus, $5 = 2x - 1$ and so $2x = 6$ or $x = 3$. This means that $x = 3$ is the only solution.
(We can substitute $x = 3$ into the original equation to verify that this is indeed a solution.)

(b) The sum of the entries in the second column is $20 + 4 + (-12) = 12$.
This means that the sum of the entries in each row, in each column, and on each diagonal is 12.
In the first row, we have $0 + 20 + a = 12$ and so $a = -8$.
On the "top left to bottom right" diagonal, we have $0 + 4 + b = 12$ and so $b = 8$.
In the third column, we have entries $a = -8$ and $b = 8$ whose sum is 0. Thus, the third entry must be 12.
Finally, in the second row, we have $c + 4 + 12 = 12$ and so $c = -4$.
In summary, $a = -8$, $b = 8$, and $c = -4$.
We can complete the magic square to obtain:

0	20	−8
−4	4	12
16	−12	8

(c) (i) If $100^2 - n^2 = 9559$, then $n^2 = 100^2 - 9559 = 10\,000 - 9559 = 441$.
Since $n > 0$, then $n = \sqrt{441} = 21$.

(ii) From (i), $9559 = 100^2 - 21^2$.
Factoring the right side as a difference of squares, we see that

$$9559 = (100 + 21)(100 - 21) = 121 \cdot 79$$

Therefore, $(a, b) = (121, 79)$ satisfies the conditions.
(In addition, the pairs $(a, b) = (79, 121), (869, 11), (11, 869)$ satisfy the conditions. The last two of these pairs cannot be obtained in the same way.)

3. (a) The area of quadrilateral $ABCD$ is the sum of the areas of $\triangle ABC$ and $\triangle ACD$.
Since $\triangle ABC$ is right-angled at B, its area equals $\frac{1}{2}(AB)(BC) = \frac{1}{2}(3)(4) = 6$.
Since $\triangle ABC$ is right-angled at B, then by the Pythagorean Theorem,

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

because $AC > 0$. (We could have also observed that $\triangle ABC$ must be a "3-4-5" triangle.)
Since $\triangle ACD$ is right-angled at A, then by the Pythagorean Theorem,

$$AD = \sqrt{CD^2 - AC^2} = \sqrt{13^2 - 5^2} = \sqrt{144} = 12$$

because $AD > 0$. (We could have also observed that $\triangle ACD$ must be a "5-12-13" triangle.)
Thus, the area of $\triangle ACD$ equals $\frac{1}{2}(AC)(AD) = \frac{1}{2}(5)(12) = 30$.
Finally, the area of quadrilateral $ABCD$ is thus $6 + 30 = 36$.

(b) Let the width of each of the identical rectangles be a.
In other words, $QP = RS = TW = WX = UV = VY = a$.
Let the height of each of the identical rectangles be b.
In other words, $QR = PS = TU = WV = XY = b$.
The perimeter of the whole shape equals

$$QP + PS + SX + XY + VY + UV + TU + TR + QR$$

Substituting for known lengths, we obtain

$$a + b + SX + b + a + a + b + TR + b$$

or $3a + 4b + (SX + TR)$.
But $SX + TR = (TR + RS + SX) - RS = (TW + WX) - RS = a + a - a = a$.
Therefore, the perimeter of the whole shape equals $4a + 4b$.
The perimeter of one rectangle is $2a + 2b$, which we are told equals 21 cm.
Finally, the perimeter of the whole shape is thus $2(2a + 2b)$ which equals 42 cm.

(c) *Solution 1*
Suppose that the rectangular prism has dimensions a cm by b cm by c cm.
Suppose further that one of the faces that is a cm by b cm is the face with area 27 cm^2 and that one of the faces that is a cm by c cm is the face with area 32 cm^2. (Since every pair of non-congruent faces shares exactly one side length, there is no loss of generality in picking these particular variables for these faces.)
Therefore, $ab = 27$ and $ac = 32$.
Further, we are told that the volume of the prism is 144 cm^3, and so $abc = 144$.

Thus, $bc = \dfrac{a^2b^2c^2}{a^2bc} = \dfrac{(abc)^2}{(ab)(ac)} = \dfrac{144^2}{(27)(32)} = 24$.
(We could also note that $abc = 144$ means $a^2b^2c^2 = 144^2$ or $(ab)(ac)(bc) = 144^2$ and so $bc = \dfrac{144^2}{(27)(32)}$.)
In other words, the third type of face of the prism has area 24 cm^2.
Thus, since the prism has two faces of each type, the surface area of the prism is equal to $2(27\text{ cm}^2 + 32\text{ cm}^2 + 24\text{ cm}^2)$ or 166 cm^2.

Solution 2
Suppose that the rectangular prism has dimensions a cm by b cm by c cm.
Suppose further that one of the faces that is a cm by b cm is the face with area 27 cm^2 and that one of the faces that is a cm by c cm is the face with area 32 cm^2. (Since every pair of non-congruent faces shares exactly one side length, there is no loss of generality in picking these particular variables for these faces.)
Therefore, $ab = 27$ and $ac = 32$.
Further, we are told that the volume of the prism is 144 cm^3, and so $abc = 144$.
Since $abc = 144$ and $ab = 27$, then $c = \frac{144}{27} = \frac{16}{3}$.
Since $abc = 144$ and $ac = 32$, then $b = \frac{144}{32} = \frac{9}{2}$.
This means that $bc = \frac{16}{3} \cdot \frac{9}{2} = 24$.
In cm^2, the surface area of the prism equals $2ab + 2ac + 2bc = 2(27) + 2(32) + 2(24) = 166$.
Thus, the surface area of the prism is 166 cm^2.

4. (a) *Solution 1*
We expand the right sides of the two equations, collecting like terms in each case:

$$y = a(x-2)(x+4) = a(x^2+2x-8) = ax^2+2ax-8a$$
$$y = 2(x-h)^2+k = 2(x^2-2hx+h^2)+k = 2x^2-4hx+(2h^2+k)$$

Since these two equations represent the same parabola, then the corresponding coefficients must be equal. That is, $a = 2$ and $2a = -4h$ and $-8a = 2h^2 + k$.
Since $a = 2$ and $2a = -4h$, then $4 = -4h$ and so $h = -1$.
Since $-8a = 2h^2 + k$ and $a = 2$ and $h = -1$, then $-16 = 2 + k$ and so $k = -18$.
Thus, $a = 2$, $h = -1$, and $k = -18$.

Solution 2
From the equation $y = a(x-2)(x+4)$, we can find the axis of symmetry by calculating the midpoint of the x-intercepts.
Since the x-intercepts are 2 and -4, the axis of symmetry is at $x = \frac{1}{2}(2 + (-4)) = -1$.
Since the vertex of the parabola lies on the axis of symmetry, then the x-coordinate of the vertex is -1.
To find the y-coordinate of the vertex, we substitute $x = -1$ back into the equation $y = a(x-2)(x+4)$ to obtain $y = a(-1-2)(-1+4) = -9a$.
Thus, the vertex of the parabola is $(-1, -9a)$.
Since the second equation for the same parabola is in vertex form, $y = 2(x-h)^2 + k$, we can see that the vertex is at (h, k) and $a = 2$.
Since $a = 2$, the vertex has coordinates $(-1, -18)$, which means that $h = -1$ and $k = -18$.
Thus, $a = 2$, $h = -1$ and $k = -18$.

(b) Let the common difference in this arithmetic sequence be d.
Since the first term in the sequence is 5, then the 5 terms are $5, 5+d, 5+2d, 5+3d, 5+4d$.
From the given information, $5^2 + (5+d)^2 + (5+2d)^2 = (5+3d)^2 + (5+4d)^2$.
Manipulating, we obtain the following equivalent equations:

$$\begin{aligned}
5^2 + (5+d)^2 + (5+2d)^2 &= (5+3d)^2 + (5+4d)^2 \\
25 + (25+10d+d^2) + (25+20d+4d^2) &= (25+30d+9d^2) + (25+40d+16d^2) \\
75 + 30d + 5d^2 &= 50 + 70d + 25d^2 \\
0 &= 20d^2 + 40d - 25 \\
0 &= 4d^2 + 8d - 5 \\
0 &= (2d+5)(2d-1)
\end{aligned}$$

Therefore, $d = -\frac{5}{2}$ or $d = \frac{1}{2}$.
These give possible fifth terms of $5 + 4d = 5 + 4(-\frac{5}{2}) = -5$ and $5 + 4d = 5 + 4(\frac{1}{2}) = 7$.
(We note that, for these two values of d, the sequences are $5, \frac{5}{2}, 0, -\frac{5}{2}, -5$ and $5, \frac{11}{2}, 6, \frac{13}{2}, 7$.)

5. (a) First, we determine the perfect squares between 1300 and 1400 and between 1400 and 1500.
Since $\sqrt{1300} \approx 36.06$, then the first perfect square larger than 1300 is $37^2 = 1369$.
The next perfect squares are $38^2 = 1444$ and $39^2 = 1521$.
Since Dan was born between 1300 and 1400 in a year that was a perfect square, then Dan was born in 1369.
Since Steve was born between 1400 and 1500 in a year that was a perfect square, then Steve was born in 1444.
Suppose that on April 7 in some year, Dan was m^2 years old and Steve was n^2 years old for some positive integers m and n. Thus, Dan was m^2 years old in the year $1369 + m^2$ and Steve was n^2 years old in the year $1444 + n^2$.
Since these represent the same years, then $1369 + m^2 = 1444 + n^2$, or $m^2 - n^2 = 1444 - 1369 = 75$.
In other words, we want to find two perfect squares less than 110 (since their ages are less than 110) whose difference is 75.
The perfect squares less than 110 are $1, 4, 9, 16, 25, 36, 49, 64, 81, 100$.
The two that differ by 75 are 100 and 25.
Thus, $m^2 = 100$ and $n^2 = 25$.
This means that the year in which the age of each of Dan and Steve was a perfect square was the year $1369 + 100 = 1469$.

(b) *Solution 1*
$\triangle ABC$ is right-angled exactly when one of the following statements is true:

- AB is perpendicular to BC, or
- AB is perpendicular to AC, or
- AC is perpendicular to BC.

Since $A(1,2)$ and $B(11,2)$ share a y-coordinate, then AB is horizontal.
For AB and BC to be perpendicular, BC must be vertical.
Thus, $B(11,2)$ and $C(k,6)$ must have the same x-coordinate, and so $k = 11$.
For AB and AC to be perpendicular, AC must be vertical.
Thus, $A(1,2)$ and $C(k,6)$ must have the same x-coordinate, and so $k = 1$.

For AC to be perpendicular to BC, their slopes must have a product of -1.
The slope of AC is $\dfrac{6-2}{k-1}$, which equals $\dfrac{4}{k-1}$.
The slope of BC is $\dfrac{6-2}{k-11}$, which equals $\dfrac{4}{k-11}$.
Thus, AC and BC are perpendicular when $\dfrac{4}{k-1}\cdot\dfrac{4}{k-11}=-1$.
Assuming that $k\neq 1$ and $k\neq 11$, we manipulate to obtain $16=-(k-1)(k-11)$ or $16=-k^2+12k-11$ or $k^2-12k+27=0$.
Factoring, we obtain $(k-3)(k-9)=0$ and so AC and BC are perpendicular when $k=3$ or $k=9$.

In summary, $\triangle ABC$ is right-angled when k equals one of $1,3,9,11$.

Solution 2
$\triangle ABC$ is right-angled exactly when its three side lengths satisfy the Pythagorean Theorem in some orientation. That is, $\triangle ABC$ is right-angled exactly when $AB^2+BC^2=AC^2$ or $AB^2+AC^2=BC^2$ or $AC^2+BC^2=AB^2$.
Using $A(1,2)$ and $B(11,2)$, we obtain $AB^2=(11-1)^2+(2-2)^2=100$.
Using $A(1,2)$ and $C(k,6)$, we obtain $AC^2=(k-1)^2+(6-2)^2=(k-1)^2+16$.
Using $B(11,2)$ and $C(k,6)$, we obtain $BC^2=(k-11)^2+(6-2)^2=(k-11)^2+16$.
Using the Pythagorean relationships above, $\triangle ABC$ is right-angled when one of the following is true:

(i)

$$\begin{aligned}100+((k-11)^2+16)&=(k-1)^2+16\\100+k^2-22k+121+16&=k^2-2k+1+16\\220&=20k\\k&=11\end{aligned}$$

(ii)

$$\begin{aligned}100+((k-1)^2+16)&=(k-11)^2+16\\100+k^2-2k+1+16&=k^2-22k+121+16\\20k&=20\\k&=1\end{aligned}$$

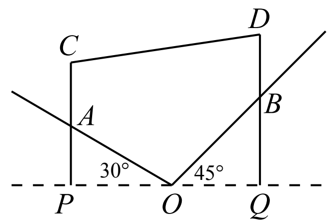
(iii)

$$\begin{aligned}((k-1)^2+16)+((k-11)^2+16)&=100\\k^2-2k+1+16+k^2-22k+121+16&=100\\2k^2-24k+54&=0\\k^2-12k+27&=0\\(k-3)(k-9)&=0\end{aligned}$$

and so $k=3$ or $k=9$.

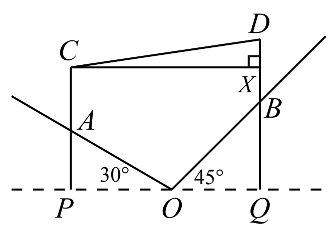
In summary, $\triangle ABC$ is right-angled when k equals one of $1,3,9,11$.

6. (a) Extend CA and DB downwards until they meet the horizontal through O at P and Q, respectively.



Since CA and DB are vertical, then $\angle CPO = \angle DQO = 90^\circ$.
Since $OA = 20$ m, then $AP = OA \sin 30^\circ = (20 \text{ m}) \cdot \frac{1}{2} = 10$ m.
Since $OB = 20$ m, then $BQ = OB \sin 45^\circ = (20 \text{ m}) \cdot \frac{1}{\sqrt{2}} = 10\sqrt{2}$ m.
Since $AC = 6$ m, then $CP = AC + AP = 16$ m.
For CD to be as short as possible and given that C is fixed, then it must be the case that CD is horizontal:

If CD were not horizontal, then suppose that X is on DQ, possibly extended, so that CX is horizontal.



Then $\angle CXD = 90^\circ$ and so $\triangle CXD$ is right-angled with hypotenuse CD.
In this case, CD is longer than CX or XD.
In particular, $CD > CX$, which means that if D were at X, then CD would be shorter.
In other words, a horizontal CD makes CD as short as possible.

When CD is horizontal, $CDQP$ is a rectangle, since it has two vertical and two horizontal sides. Thus, $DQ = CP = 16$ m.
Finally, this means that $BD = DQ - BQ = (16 - 10\sqrt{2})$ m.

(b) Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, then we assume that $\cos\theta \neq 0$.
Therefore, we obtain the following equivalent equations:

$$\begin{aligned}
\cos\theta &= \tan\theta \\
\cos\theta &= \frac{\sin\theta}{\cos\theta} \\
\cos^2\theta &= \sin\theta \\
1 - \sin^2\theta &= \sin\theta \\
0 &= \sin^2\theta + \sin\theta - 1
\end{aligned}$$

Let $u = \sin\theta$. This quadratic equation becomes $u^2 + u - 1 = 0$.

By the quadratic formula, $u = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \dfrac{-1 \pm \sqrt{5}}{2}$.

Therefore, $\sin\theta = \dfrac{-1 + \sqrt{5}}{2} \approx 0.62$ or $\sin\theta = \dfrac{-1 - \sqrt{5}}{2} \approx -1.62$.

Since $-1 \leq \sin\theta \leq 1$, then the second solution is inadmissible. Thus, $\sin\theta = \dfrac{-1 + \sqrt{5}}{2}$.

7. (a) *Solution 1*

Suppose that the trains are travelling at v km/h.

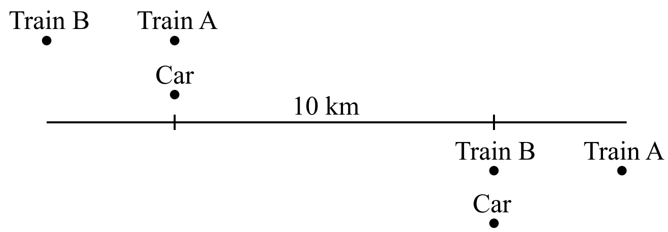
Consider two consecutive points in time at which the car is passed by a train.

Since these points are 10 minutes apart, and 10 minutes equals $\frac{1}{6}$ hour, and the car travels at 60 km/h, then the car travels $(60 \text{ km/h}) \cdot (\frac{1}{6} \text{ h}) = 10$ km.

During these 10 minutes, each train travels $\frac{1}{6}v$ km, since its speed is v km/h.

At the first instance, Train A and the car are next to each other.

At this time, Train B is "3 minutes" behind Train A.



Since 3 minutes is $\frac{1}{20}$ hour, then Train B is $\frac{1}{20}v$ km behind Train A and the car.

Therefore, the distance from the location of Train B at the first instance to the location where it passes the car is $(\frac{1}{20}v + 10)$ km.

But this distance also equals $\frac{1}{6}v$ km, since Train B travels for 10 minutes.

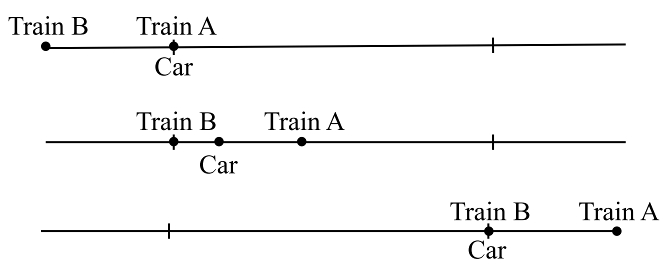
Thus, $\frac{1}{6}v = \frac{1}{20}v + 10$ or $\frac{10}{60}v - \frac{3}{60}v = 10$ and so $\frac{7}{60}v = 10$ or $v = \frac{600}{7}$.

Therefore, the trains are travelling at $\frac{600}{7}$ km/h.

Solution 2

Suppose that the trains are travelling at v km/h.

Consider the following three points in time: the instant when the car and Train A are next to each other, the instant when Train B is at the same location that the car and Train A were at in the previous instant, and the instant when the car and Train B are next to each other.



From the first instant to the second, Train B "catches up" to where Train A was, so this must take a total of 3 minutes, because the trains leave the station 3 minutes apart.

Since 3 minutes equals $\frac{3}{60}$ hour and the car travels at 60 km/h, then the car travels $(60 \text{ km/h}) \cdot (\frac{3}{60} \text{ h}) = 3$ km between these two instants.

From the first instant to the third, 10 minutes passes, since these are consecutive points at which the car is passed by trains. In 10 minutes, the car travels 10 km.

Therefore, between the second and third instants, $10 - 3 = 7$ minutes pass. During these 7 minutes, Train B travels 10 km.

Since 7 minutes equals $\frac{7}{60}$ hour, then $v \text{ km/h} = \frac{10 \text{ km}}{7/60 \text{ h}} = \frac{600}{7}$ km/h, and so the trains are travelling at $\frac{600}{7}$ km/h.

(b) From the first equation, we note that $a \geq 0$ and $b \geq 0$, since the argument of a square root must be non-negative.
From the second equation, we note that $a > 0$ and $b > 0$, since the argument of a logarithm must be positive.
Combining these restrictions, we see that $a > 0$ and $b > 0$.

From the equation $\log_{10} a + \log_{10} b = 2$, we obtain $\log_{10}(ab) = 2$ and so $ab = 10^2 = 100$.
From the first equation, obtain

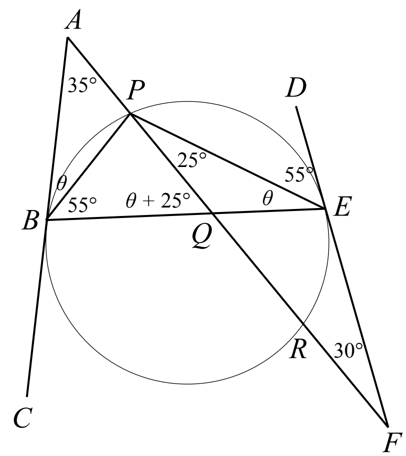
$$\begin{aligned}
(\sqrt{a} + \sqrt{b})^2 &= 8^2 \\
a + 2\sqrt{ab} + b &= 64 \\
a + 2\sqrt{100} + b &= 64 \\
a + b &= 64 - 2\sqrt{100} = 44
\end{aligned}$$

Since $a + b = 44$, then $b = 44 - a$.
Since $ab = 100$, then $a(44 - a) = 100$ or $44a - a^2 = 100$ and so $0 = a^2 - 44a + 100$.
By the quadratic formula,

$$a = \frac{44 \pm \sqrt{44^2 - 4(1)(100)}}{2 \cdot 1} = \frac{44 \pm \sqrt{1536}}{2} = \frac{44 \pm 16\sqrt{6}}{2} = 22 \pm 8\sqrt{6}$$

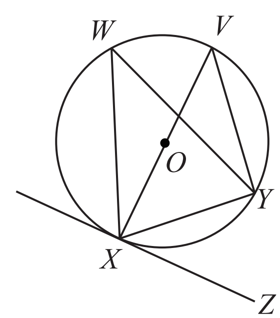
Since $b = 44 - a$, then $b = 44 - (22 \pm 8\sqrt{6}) = 22 \mp 8\sqrt{6}$.
Therefore, $(a, b) = (22 + 8\sqrt{6}, 22 - 8\sqrt{6})$ or $(a, b) = (22 - 8\sqrt{6}, 22 + 8\sqrt{6})$.
(We note that $22 + 8\sqrt{6} > 0$ and $22 - 8\sqrt{6} > 0$, so the initial restrictions on a and b are satisfied.)

8. (a) Let $\angle PEQ = \theta$.
Join P to B.
We use the fact that the angle between a tangent to a circle and a chord in that circle that passes through the point of tangency equals the angle inscribed by that chord. We prove this fact below.
More concretely, $\angle DEP = \angle PBE$ (using the chord EP and the tangent through E) and $\angle ABP = \angle PEQ = \theta$ (using the chord BP and the tangent through B).
Now $\angle DEP$ is exterior to $\triangle FEP$ and so $\angle DEP = \angle FPE + \angle EFP = 25^\circ + 30^\circ$, and so $\angle PBE = \angle DEP = 55^\circ$.
Furthermore, $\angle AQB$ is an exterior angle of $\triangle PQE$.
Thus, $\angle AQB = \angle QPE + \angle PEQ = 25^\circ + \theta$.



In $\triangle ABQ$, we have $\angle BAQ = 35^\circ$, $\angle ABQ = \theta + 55^\circ$, and $\angle AQB = 25^\circ + \theta$.
Thus, $35^\circ + (\theta + 55^\circ) + (25^\circ + \theta) = 180^\circ$ or $115^\circ + 2\theta = 180^\circ$, and so $2\theta = 65^\circ$.
Therefore $\angle PEQ = \theta = \frac{1}{2}(65^\circ) = 32.5^\circ$.

As an addendum, we prove that the angle between a tangent to a circle and a chord in that circle that passes through the point of tangency equals the angle inscribed by that chord.

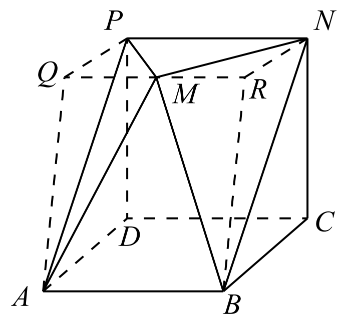


Consider a circle with centre O and a chord XY, with tangent ZX meeting the circle at X. We prove that if ZX is tangent to the circle, then $\angle ZXY$ equals $\angle XWY$ whenever W is a point on the circle on the opposite side of XY as XZ (that is, the angle subtended by XY on the opposite side of the circle).
We prove this in the case that $\angle ZXY$ is acute. The cases where $\angle ZXY$ is a right angle or an obtuse angle are similar.
Draw diameter XOV and join VY.
Since $\angle ZXY$ is acute, points V and W are on the same arc of chord XY.
This means that $\angle XVY = \angle XWY$, since they are angles subtended by the same chord.
Since OX is a radius and XZ is a tangent, then $\angle OXZ = 90^\circ$.
Thus, $\angle OXY + \angle ZXY = 90^\circ$.
Since XV is a diameter, then $\angle XYV = 90^\circ$.
From $\triangle XYV$, we see that $\angle XVY + \angle VXY = 90^\circ$.
But $\angle OXY + \angle ZXY = 90^\circ$ and $\angle XVY + \angle VXY = 90^\circ$ and $\angle OXY = \angle VXY$ tells us that $\angle ZXY = \angle XVY$.
This gives us that $\angle ZXY = \angle XWY$, as required.

(b) *Solution 1*

Draw a line segment through M in the plane of $\triangle PMN$ parallel to PN and extend this line until it reaches the plane through P, A and D at Q on one side and the plane through N, B and C at R on the other side.

Join Q to P and A. Join R to N and B.



So the volume of solid $ABCDPMN$ equals the volume of solid $ABCDPQRN$ minus the volumes of solids $PMQA$ and $NMRB$.

Solid $ABCDPQRN$ is a trapezoidal prism. This is because NR and BC are parallel (since they lie in parallel planes), which makes $NRBC$ a trapezoid. Similarly, $PQAD$ is a trapezoid. Also, PN, QR, DC, and AB are all perpendicular to the planes of these trapezoids and equal in length, since they equal the side lengths of the squares.

Solids $PMQA$ and $NMRB$ are triangular-based pyramids. We can think of their bases as being $\triangle PMQ$ and $\triangle NMR$. Their heights are each equal to 2, the height of the original solid. (The volume of a triangular-based pyramid equals $\frac{1}{3}$ times the area of its base times its height.)

The volume of $ABCDPQRN$ equals the area of trapezoid $NRBC$ times the width of the prism, which is 2.

That is, this volume equals $\frac{1}{2}(NR + BC)(NC)(NP) = \frac{1}{2}(NR + 2)(2)(2) = 2 \cdot NR + 4$.

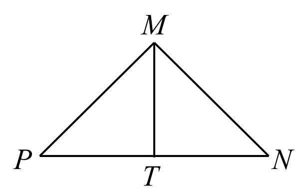
So we need to find the length of NR.

Consider quadrilateral $PNRQ$. This quadrilateral is a rectangle since PN and QR are perpendicular to the two side planes of the original solid.

Thus, NR equals the height of $\triangle PMN$.

Join M to the midpoint T of PN.

Since $\triangle PMN$ is isosceles, then MT is perpendicular to PN.



Since $NT = \frac{1}{2}PN = 1$ and $\angle PMN = 90^\circ$ and $\angle TNM = 45^\circ$, then $\triangle MTN$ is also right-angled and isosceles with $MT = TN = 1$.

Therefore, $NR = MT = 1$ and so the volume of $ABCDPQRN$ is $2 \cdot 1 + 4 = 6$.

The volumes of solids $PMQA$ and $NMRB$ are equal. Each has height 2 and their bases $\triangle PMQ$ and $\triangle NMR$ are congruent, because each is right-angled (at Q and at R) with $PQ = NR = 1$ and $QM = MR = 1$.

Thus, using the formula above, the volume of each is $\frac{1}{3}(\frac{1}{2}(1)(1))2 = \frac{1}{3}$.

Finally, the volume of the original solid equals $6 - 2 \cdot \frac{1}{3} = \frac{16}{3}$.

Solution 2

We determine the volume of $ABCDPMN$ by splitting it into two solids: $ABCDPN$ and $ABNPM$ by slicing along the plane of $ABNP$.

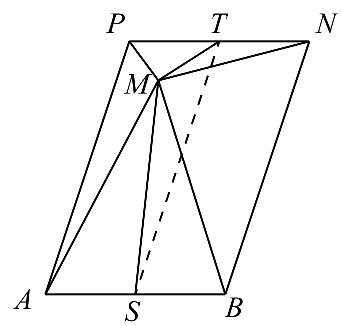
Solid $ABCDPN$ is a triangular prism, since $\triangle BCN$ and $\triangle ADP$ are each right-angled (at C and D), $BC = CN = AD = DP = 2$, and segments PN, DC and AB are perpendicular to each of the triangular faces and equal in length.

Thus, the volume of $ABCDPN$ equals the area of $\triangle BCN$ times the length of DC, or $\frac{1}{2}(BC)(CN)(DC) = \frac{1}{2}(2)(2)(2) = 4$. (This solid can also be viewed as "half" of a cube.)

Solid $ABNPM$ is a pyramid with rectangular base $ABNP$. (Note that PN and AB are perpendicular to the planes of both of the side triangular faces of the original solid, that $PN = AB = 2$ and $BN = AP = \sqrt{2^2 + 2^2} = 2\sqrt{2}$, by the Pythagorean Theorem.)

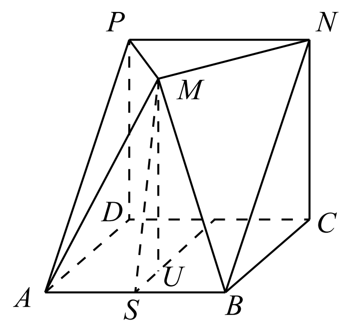
Therefore, the volume of $ABNPM$ equals $\frac{1}{3}(AB)(BN)h = \frac{4\sqrt{2}}{3}h$, where h is the height of the pyramid (that is, the distance that M is above plane $ABNP$).

So we need to calculate h.



Join M to the midpoint, T, of PN and to the midpoint, S, of AB. Join S and T. By symmetry, M lies directly above ST. Since $ABNP$ is a rectangle and S and T are the midpoints of opposite sides, then $ST = AP = 2\sqrt{2}$.

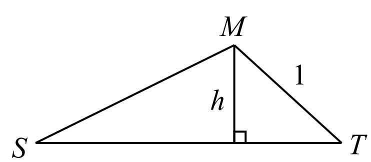
Since $\triangle PMN$ is right-angled and isosceles, then MT is perpendicular to PN. Since $NT = \frac{1}{2}PN = 1$ and $\angle TNM = 45^\circ$, then $\triangle MTN$ is also right-angled and isosceles with $MT = TN = 1$.



Also, MS is the hypotenuse of the triangle formed by dropping a perpendicular from M to U in the plane of $ABCD$ (a distance of 2) and joining U to S. Since M is 1 unit horizontally from PN, then $US = 1$.

Thus, $MS = \sqrt{2^2 + 1^2} = \sqrt{5}$ by the Pythagorean Theorem.

We can now consider $\triangle SMT$. h is the height of this triangle, from M to base ST.



Now $h = MT \sin(\angle MTS) = \sin(\angle MTS)$.

By the cosine law in $\triangle SMT$, we have

$$MS^2 = ST^2 + MT^2 - 2(ST)(MT)\cos(\angle MTS)$$

Therefore, $5 = 8 + 1 - 4\sqrt{2}\cos(\angle MTS)$ or $4\sqrt{2}\cos(\angle MTS) = 4$.

Thus, $\cos(\angle MTS) = \frac{1}{\sqrt{2}}$ and so $\angle MTS = 45^\circ$ which gives $h = \sin(\angle MTS) = \frac{1}{\sqrt{2}}$.

(Alternatively, we note that the plane of $ABCD$ is parallel to the plane of PMN, and so since the angle between plane $ABCD$ and plane $PNBA$ is 45°, then the angle between plane $PNBA$ and plane PMN is also 45°, and so $\angle MTS = 45^\circ$.)

Finally, this means that the volume of $ABNPM$ is $\frac{4\sqrt{2}}{3} \cdot \frac{1}{\sqrt{2}} = \frac{4}{3}$, and so the volume of solid $ABCDPMN$ is $4 + \frac{4}{3} = \frac{16}{3}$.

9. (a) There are $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ permutations of $1, 2, 3, 4$.
This is because there are 4 possible choices for a_1, and for each of these there are 3 possible choices for a_2, and for each of these there are 2 possible choices for a_3, and then 1 possible choice for a_4.
Consider the permutation $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4$. (We write this as 1,2,3,4.)
Here, $|a_1 - a_2| + |a_3 - a_4| = |1 - 2| + |3 - 4| = 1 + 1 = 2$.
This value is the same as the value for each of 2,1,3,4 and 1,2,4,3 and 2,1,4,3 and 3,4,1,2 and 4,3,1,2 and 3,4,2,1 and 4,3,2,1.
Consider the permutation 1,3,2,4.
Here, $|a_1 - a_2| + |a_3 - a_4| = |1 - 3| + |2 - 4| = 2 + 2 = 4$.
This value is the same as the value for each of 3,1,2,4 and 1,3,4,2 and 3,1,4,2 and 2,4,1,3 and 4,2,1,3 and 2,4,3,1 and 4,2,3,1.
Consider the permutation 1,4,2,3.
Here, $|a_1 - a_2| + |a_3 - a_4| = |1 - 4| + |2 - 3| = 3 + 1 = 4$.
This value is the same as the value for each of 4,1,2,3 and 1,4,3,2 and 4,1,3,2 and 2,3,1,4 and 3,2,1,4 and 2,3,4,1 and 3,2,4,1.
This accounts for all 24 permutations.
Therefore, the average value is $\dfrac{2 \cdot 8 + 4 \cdot 8 + 4 \cdot 8}{24} = \dfrac{80}{24} = \dfrac{10}{3}$.

(b) There are $7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ permutations of $1, 2, 3, 4, 5, 6, 7$, because there are 7 choices for a_1, then 6 choices for a_2, and so on.
We determine the average value of $a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7$ over all of these permutations by determining the sum of all 7! values of this expression and dividing by 7!.
To determine the sum of all 7! values, we determine the sum of the values of a_1 in each of these expressions and call this total s_1, the sum of the values of a_2 in each of these expressions and call this total s_2, and so on.
The sum of the 7! values of the original expression must equal $s_1 - s_2 + s_3 - s_4 + s_5 - s_6 + s_7$.
This uses the fact that, when adding, the order in which we add the same set of numbers does not matter.
By symmetry, the sums of the values of $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ will all be equal. That is, $s_1 = s_2 = s_3 = s_4 = s_5 = s_6 = s_7$.
This means that the desired average value equals

$$\frac{s_1 - s_2 + s_3 - s_4 + s_5 - s_6 + s_7}{7!} = \frac{(s_1 + s_3 + s_5 + s_7) - (s_2 + s_4 + s_6)}{7!} = \frac{4s_1 - 3s_1}{7!} = \frac{s_1}{7!}$$

So we need to determine the value of s_1.
Now a_1 can equal each of $1, 2, 3, 4, 5, 6, 7$.
If $a_1 = 1$, there are 6! combinations of values for $a_2, a_3, a_4, a_5, a_6, a_7$, since there are still 6 choices for a_2, 5 for a_3, and so on.
Similarly, there are 6! combinations with a_1 equal to each of $2, 3, 4, 5, 6, 7$.
Thus, $s_1 = 1 \cdot 6! + 2 \cdot 6! + 3 \cdot 6! + 4 \cdot 6! + 5 \cdot 6! + 6 \cdot 6! + 7 \cdot 6! = 6!(1+2+3+4+5+6+7) = 28(6!)$.
Therefore, the average value of the expression is $\dfrac{28(6!)}{7!} = \dfrac{28(6!)}{7(6!)} = \dfrac{28}{7} = 4$.

(c) There are 200! permutations of $1, 2, 3, \ldots, 198, 199, 200$.
We determine the average value of

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_{197} - a_{198}| + |a_{199} - a_{200}| \qquad (*)$$

over all of these permutations by determining the sum of all 200! values of this expression and dividing by 200!.
As in (b), we let s_1 be the sum of the values of $|a_1 - a_2|$ in each of these expressions, s_2 be the sum of the values of $|a_3 - a_4|$, and so on.
The sum of the 200! values of $(*)$ equals $s_1 + s_2 + \cdots + s_{99} + s_{100}$.
By symmetry, $s_1 = s_2 = \cdots = s_{99} = s_{100}$.
Therefore, the average value of $(*)$ equals $\dfrac{100s_1}{200!}$. So we need to determine the value of s_1.
Suppose that $a_1 = i$ and $a_2 = j$ for some integers i and j between 1 and 200, inclusive.
There are 198! permutations with $a_1 = i$ and $a_2 = j$ because there are still 198 choices for a_3, 197 choices for a_4, and so on.
Similarly, there are 198! permutations with $a_1 = j$ and $a_2 = i$.
Since $|i - j| = |j - i|$, then there are 2(198!) permutations with $|a_1 - a_2| = |i - j|$ that come from a_1 and a_2 equalling i and j in some order.
Therefore, we may assume that $i > j$ and note that s_1 equals 2(198!) times the sum of $i - j$ over all possible pairs $i > j$.
(Note that there are $\dbinom{200}{2} = \dfrac{200(199)}{2}$ choices for the pair of integers (i, j) with $i > j$. For each of these choices, there are 2(198!) choices for the remaining entries in the permutation, which gives $\dfrac{200(199)}{2} \cdot 2(198!) = 200(199)(198!) = 200!$ permutations, as expected.)
So to determine s_1, we need to determine the sum of the values of $i - j$.
We calculate this sum, which we call D, by letting $j = 1, 2, 3, \ldots, 198, 199$ and for each of these, we let i be the possible integers with $j < i \leq 200$:

$$\begin{aligned}
D = &(2-1) + (3-1) + (4-1) + \cdots + (197-1) + (198-1) + (199-1) + (200-1) \\
&+ (3-2) + (4-2) + (5-2) + \cdots + (198-2) + (199-2) + (200-2) \\
&+ (4-3) + (5-3) + (6-3) + \cdots + (199-3) + (200-3) \\
&\vdots \\
&+ (199-198) + (200-198) \\
&+ (200-199) \\
&= 199(1) + 198(2) + 197(3) + \cdots + 2(198) + 1(199) \qquad \text{(grouping by columns)} \\
&= 199(200-199) + 198(200-198) + 197(200-197) + \cdots + 2(200-2) + 1(200-1) \\
&= 200(199 + 198 + 197 + \cdots + 3 + 2 + 1) - (199^2 + 198^2 + 197^2 + \cdots + 3^2 + 2^2 + 1^2) \\
&= 200 \cdot \tfrac{1}{2}(199)(200) - \tfrac{1}{6}(199)(199+1)(2(199)+1) \\
&= 100(199)(200) - \tfrac{1}{6}(199)(200)(399) \\
&= 199(200)\left(100 - \tfrac{133}{2}\right) \\
&= 199(200)\tfrac{67}{2}
\end{aligned}$$

Therefore, $s_1 = 2(198!)D = 2(198!) \cdot \frac{199(200)(67)}{2} = 67(198!)(199)(200) = 67(200!)$.
Finally, this means that the average value of $(*)$ is $\dfrac{100s_1}{200!} = \dfrac{100(67)(200!)}{200!} = 6700$.

We note that we have used the facts that, if n is a positive integer, then

- $1 + 2 + \cdots + (n-1) + n = \frac{1}{2}n(n+1)$
- $1^2 + 2^2 + \cdots + (n-1)^2 + n^2 = \frac{1}{6}n(n+1)(2n+1)$

Using sigma notation, we could have calculated D as follows:

$$\begin{aligned}
D &= \sum_{i=2}^{200} \sum_{j=1}^{i-1} (i-j) \\
&= \left(\sum_{i=2}^{200} \sum_{j=1}^{i-1} i \right) - \left(\sum_{i=2}^{200} \sum_{j=1}^{i-1} j \right) \\
&= \left(\sum_{i=2}^{200} i(i-1) \right) - \left(\sum_{i=2}^{200} \tfrac{1}{2}(i-1)i \right) \\
&= \left(\sum_{i=2}^{200} i(i-1) \right) - \tfrac{1}{2} \left(\sum_{i=2}^{200} (i-1)i \right) \\
&= \tfrac{1}{2} \left(\sum_{i=2}^{200} (i-1)i \right) \\
&= \tfrac{1}{2} \left(\sum_{i=1}^{200} (i-1)i \right) \qquad \text{(since } (i-1)i = 0 \text{ when } i = 1\text{)} \\
&= \tfrac{1}{2} \left(\sum_{i=1}^{200} (i^2 - i) \right) \\
&= \tfrac{1}{2} \left(\sum_{i=1}^{200} i^2 - \sum_{i=1}^{200} i \right) \\
&= \tfrac{1}{2} \left(\tfrac{1}{6}(200)(200+1)(2(200)+1) - \tfrac{1}{2}(200)(200+1) \right) \\
&= \tfrac{1}{2}(200)(201) \left(\tfrac{1}{6}(401) - \tfrac{1}{2} \right) \\
&= 100(201) \cdot \tfrac{398}{6} \\
&= 100(201) \cdot \tfrac{199}{3} \\
&= 100(67)(199)
\end{aligned}$$

which equals $199(200)\frac{67}{2}$, as expected. (Can you determine a general formula when 200 is replaced with $2n$?)

10. (a) We start with the subset $\{1,2,3\}$.
The sums of pairs of elements are $1+2=3$ and $1+3=4$ and $2+3=5$, which are all different.
Thus, $\{1,2,3\}$ is exciting.
We proceed to include additional elements in $\{1,2,3\}$.
We cannot include 4 to create an exciting set, since if we did, we would have $1+4=2+3$, and so $\{1,2,3,4\}$ is boring.
Consider the subset $\{1,2,3,5\}$.
The sums of pairs of elements are

$$1+2=3 \qquad 1+3=4 \qquad 1+5=6 \qquad 2+3=5 \qquad 2+5=7 \qquad 3+5=8$$

which are all different.
Thus, $\{1,2,3,5\}$ is exciting.
We cannot include 6 or 7 since $2+5=1+6$ and $3+5=1+7$.
Consider the subset $\{1,2,3,5,8\}$.
In addition to the six sums above, we have the additional sums $1+8=9$ and $2+8=10$ and $3+8=11$ and $5+8=13$, so the 10 sums are all different.
Therefore, $\{1,2,3,5,8\}$ is an exciting subset of $\{1,2,3,4,5,6,7,8\}$ that contains exactly 5 elements.
(The subset $\{1,4,6,7,8\}$ is the only other exciting subset of $\{1,2,3,4,5,6,7,8\}$ that contains exactly 5 elements.)

(b) Suppose that S is an exciting set that contains exactly m elements.
There are $\binom{m}{2}=\dfrac{m(m-1)}{2}$ pairs of elements of S.
Since S is exciting, the sums of these pairs of elements are all distinct positive integers.
This means that the largest of these sums is greater than or equal to $\dfrac{m(m-1)}{2}$.
When two numbers add to $\dfrac{m(m-1)}{2}$ or greater, then at least one of them must be at least as large as $\dfrac{1}{2}\cdot\dfrac{m(m-1)}{2}=\dfrac{m^2-m}{4}$.
Therefore, there is an element of S that is greater than or equal to $\dfrac{m^2-m}{4}$.

(c) Let n be a positive integer with $n \geq 10$.
For each integer k with $1 \leq k \leq n$, define $x_k = 2n \cdot \operatorname{rem}(k^2,n)+k$, where $\operatorname{rem}(k^2,n)$ is the remainder when k^2 is divided by n.
Define $T=\{x_1,x_2,\ldots,x_{n-1},x_n\}$.
We show that T is exciting exactly when n is prime.

Suppose that a,b,c,d are distinct integers between 1 and n with $x_a+x_b=x_c+x_d$.
This equation is equivalent to

$$(2n\cdot\operatorname{rem}(a^2,n)+a)+(2n\cdot\operatorname{rem}(b^2,n)+b)=(2n\cdot\operatorname{rem}(c^2,n)+c)+(2n\cdot\operatorname{rem}(d^2,n)+d)$$

and

$$2n\cdot(\operatorname{rem}(a^2,n)+\operatorname{rem}(b^2,n)-\operatorname{rem}(c^2,n)-\operatorname{rem}(d^2,n))=c+d-a-b$$

Since a,b,c,d are distinct integers between 1 and n, inclusive, then we have $1+2\leq a+b\leq(n-1)+n$, or $3\leq a+b\leq 2n-1$. Similarly, $3\leq c+d\leq 2n-1$.

This means that $3-(2n-1) \leq c+d-a-b \leq (2n-1)-3$ or $-2n+4 \leq c+d-a-b \leq 2n-4$. But the left side of the equation

$$2n \cdot (\text{rem}(a^2, n) + \text{rem}(b^2, n) - \text{rem}(c^2, n) - \text{rem}(d^2, n)) = c + d - a - b$$

is an integer that is a multiple of $2n$, so the right side $(c + d - a - b)$ must be as well.
Since $-2n + 4 \leq c + d - a - b \leq 2n - 4$ and the only multiple of $2n$ between $-2n + 4$ and $2n - 4$ is $0 \cdot 2n = 0$, then $c + d - a - b = 0$ or $c + d = a + b$.
Thus, $2n \cdot (\text{rem}(a^2, n) + \text{rem}(b^2, n) - \text{rem}(c^2, n) - \text{rem}(d^2, n)) = 0$.
Since $n \neq 0$, then $\text{rem}(a^2, n) + \text{rem}(b^2, n) - \text{rem}(c^2, n) - \text{rem}(d^2, n) = 0$.
Therefore, $x_a + x_b = x_c + x_d$ exactly when

$$a + b = c + d \quad \text{and} \quad \text{rem}(a^2, n) + \text{rem}(b^2, n) = \text{rem}(c^2, n) + \text{rem}(d^2, n)$$

Suppose that n is composite. We show that T is boring.
We consider three cases: $n = p^2$ for some prime p, n is even, and all other n.
Suppose that $n = p^2$ for some prime p. Since $n \geq 10$, then $p \geq 5$.
Set $a = p$, $b = 4p$, $c = 2p$, and $d = 3p$.
Then $a + b = 5p = c + d$.
Also, since $p \geq 5$, then $0 < p < 2p < 3p < 4p < p^2$.
Furthermore, since each of a, b, c, d is divisible by p, then each of a^2, b^2, c^2, d^2 is divisible by $p^2 = n$, so $\text{rem}(a^2, n) = \text{rem}(b^2, n) = \text{rem}(c^2, n) = \text{rem}(d^2, n) = 0$.
This means that $a + b = c + d$ and $\text{rem}(a^2, n) + \text{rem}(b^2, n) = \text{rem}(c^2, n) + \text{rem}(d^2, n)$, and so $x_a + x_b = x_c + x_d$, which means that T is boring.

Next, suppose that n is even, say $n = 2t$ for some positive integer $t \geq 5$.
Set $a = 1$, $b = t + 2$, $c = 2$, and $d = t + 1$.
Since $t \geq 5$, then $1 \leq a < b < c < d < 2t$, so a, b, c, d are distinct positive integers in the correct range.
Also, $a + b = t + 3 = c + d$.
To show that $x_a + x_b = x_c + x_d$, it remains to show that

$$\text{rem}(1^2, 2t) + \text{rem}((t + 2)^2, 2t) = \text{rem}(2^2, 2t) + \text{rem}((t + 1)^2, 2t)$$

Now $\text{rem}(1^2, 2t) = \text{rem}(1, 2t) = 1$ and $\text{rem}(2^2, 2t) = \text{rem}(4, 2t) = 4$ since $2t > 4$.
Also, since $(t + 2)^2 = t^2 + 4t + 4$ and so $(t + 2)^2$ and $t^2 + 4$ differ by a multiple of $n = 2t$, then $\text{rem}((t + 2)^2, 2t) = \text{rem}(t^2 + 4, 2t)$.
Similarly, since $(t + 1)^2 = t^2 + 2t + 1$, then $\text{rem}((t + 1)^2, 2t) = \text{rem}(t^2 + 1, 2t)$.
Therefore, we need to show that $\text{rem}(t^2 + 4, 2t) - \text{rem}(t^2 + 1, 2t) = 4 - 1 = 3$.
Since $t \geq 5$, then $t^2 + t > t^2 + 4$.
This means that $t^2 < t^2 + 1 < t^2 + 2 < t^2 + 3 < t^2 + 4 < t^2 + t$; in other words, each of $t^2 + 1$, $t^2 + 2$, $t^2 + 3$, $t^2 + 4$ is strictly between two consecutive multiples of t, and so none of these four integers can be a multiple of t. This means that none of these is a multiple of $n = 2t$.
Therefore, $t^2 + 4$ and $t^2 + 1$ are bounded between the same two multiples of n, and so the difference between their remainders when dividing by n equals the difference between the integers, which is 3.
Thus, $x_a + x_b = x_c + x_d$, which means that T is boring.

Finally, we consider the case where n is odd and composite and can be written as $n = MN$ for some odd integers $M > N > 1$.

Set $a = \frac{1}{2}(M+N)$, $b = n - a$, $c = \frac{1}{2}(M-N)$, and $d = n - c$.
Since M and N are both odd, then $M+N$ and $M-N$ are even, and so a,b,c,d are integers.
Since $M > N > 0$, then $a > c > 0$.
Since $N \geq 3$, then $n = MN \geq 3M > 2M$ and so $M < \frac{1}{2}n$.
Since $M > N$, then $a = \frac{1}{2}(M+N) < \frac{1}{2}(M+M) = M < \frac{1}{2}n$.
Therefore, $0 < c < a < \frac{1}{2}n$.
Since $b = n - a$ and $d = n - c$, then $\frac{1}{2}n < b < d < n$ and so $0 < c < a < \frac{1}{2}n < b < d < n$.
This means that a,b,c,d are distinct integers in the correct range.
Also, note that $a + b = n = c + d$.
To show that $x_a + x_b = x_c + x_d$, it remains to show that

$$\text{rem}(a^2,n) + \text{rem}(b^2,n) = \text{rem}(c^2,n) + \text{rem}(d^2,n)$$

We show that $\text{rem}(a^2,n) = \text{rem}(b^2,n) = \text{rem}(c^2,n) = \text{rem}(d^2,n)$, which will provide the desired conclusion.
Since $b = n - a$, then $b^2 = n^2 - 2na + a^2$. Since b^2 and a^2 differ by a multiple of n, their remainders after division by n will be equal. Similarly, $\text{rem}(c^2,n) = \text{rem}(d^2,n)$.
Thus, it remains to show that $\text{rem}(a^2,n) = \text{rem}(c^2,n)$.
But

$$a^2 - c^2 = (a+c)(a-c) = \left(\tfrac{1}{2}(M+N) + \tfrac{1}{2}(M-N)\right)\left(\tfrac{1}{2}(M+N) - \tfrac{1}{2}(M-N)\right) = MN = n$$

Since a^2 and c^2 differ by a multiple of n, then $\text{rem}(a^2,n) = \text{rem}(c^2,n)$.
Thus, $x_a + x_b = x_c + x_d$, which means that T is boring.

Suppose that n is prime. We show that T is exciting.
Since $n \geq 10$, then n is odd.
Suppose that $x_a + x_b = x_c + x_d$. We will show that this is not possible.
Recall that $x_a + x_b = x_c + x_d$ is equivalent to the conditions $a + b = c + d$ and $\text{rem}(a^2,n) + \text{rem}(b^2,n) = \text{rem}(c^2,n) + \text{rem}(d^2,n)$.
We work with this second equation.
When a^2 is divided by n, we obtain a quotient that we call q_a and the remainder $\text{rem}(a^2,n)$.
Note that $a^2 = q_a n + \text{rem}(a^2,n)$ and $0 \leq \text{rem}(a^2,n) < n$.
We define q_b, q_c, q_d similarly and obtain

$$(a^2 - q_a n) + (b^2 - q_b n) = (c^2 - q_c n) + (d^2 - q_d n)$$

or

$$a^2 + b^2 - c^2 - d^2 = n(q_a + q_b - q_c - q_d)$$

Since $a + b = c + d$, then $a^2 + 2ab + b^2 = c^2 + 2cd + d^2$ or $a^2 + b^2 - c^2 - d^2 = 2cd - 2ab$.
Therefore, $x_a + x_b = x_c + x_d$ exactly when $a + b = c + d$ and $2cd - 2ab = n(q_a + q_b - q_c - q_d)$.
Since $d = a + b - c$, then this last equation becomes

$$\begin{aligned} 2c(a+b-c) - 2ab &= n(q_a + q_b - q_c - q_d) \\ -2(c^2 - ac - bc + ab) &= n(q_a + q_b - q_c - q_d) \\ -2(c(c-a) - b(c-a)) &= n(q_a + q_b - q_c - q_d) \\ -2(c-a)(c-b) &= n(q_a + q_b - q_c - q_d) \end{aligned}$$

Since $x_a + x_b = x_c + x_d$, then $a + b = c + d$ and $-2(c-a)(c-b) = n(q_a + q_b - q_c - q_d)$.
Therefore, $2(c-a)(c-b)$ is a multiple of n, which is an odd prime.

This means that either $c - a$ or $c - b$ is a multiple of n.
But a, b, c, d are between 1 and n inclusive and are distinct, so $1 - n \leq c - a \leq n - 1$ and $1 - n \leq c - b \leq n - 1$.
The only multiple of n in this range is 0, so $c - a = 0$ or $c - b = 0$, which contradicts the fact that a, b, c, d are distinct.
Therefore, if n is prime, there do not exist four distinct elements of T that make T boring, so T is exciting.

In summary, T is exciting exactly when $n \geq 10$ is prime.