The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

Euclid Contest

Thursday, April 6, 2017
(in North America and South America)

Friday, April 7, 2017
(outside of North America and South America)

UNIVERSITY OF WATERLOO

Time: $2\frac{1}{2}$ hours

Do not open this booklet until instructed to do so.

Number of questions: 10 **Each question is worth 10 marks**

Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.

Parts of each question can be of two types:

1. **SHORT ANSWER** parts indicated by
   - worth 3 marks each
   - full marks given for a correct answer which is placed in the box
   - **part marks awarded only if relevant work** is shown in the space provided
2. **FULL SOLUTION** parts indicated by
   - worth the remainder of the 10 marks for the question
   - **must be written in the appropriate location** in the answer booklet
   - marks awarded for completeness, clarity, and style of presentation
   - a correct solution poorly presented will not earn full marks

WRITE ALL ANSWERS IN THE ANSWER BOOKLET PROVIDED.

- Extra paper for your finished solutions supplied by your supervising teacher must be inserted into your answer booklet. Write your name, school name, and question number on any inserted pages.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

Do not discuss the problems or solutions from this contest online for the next 48 hours.

The name, grade, school and location, and score range of some top-scoring students will be published on our website, cemc.uwaterloo.ca. In addition, the name, grade, school and location, and score of some top-scoring students may be shared with other mathematical organizations for other recognition opportunities.

NOTE:
1. Please read the instructions on the front cover of this booklet.
2. Write all answers in the answer booklet provided.
3. For questions marked , place your answer in the appropriate box in the answer booklet and **show your work**.
4. For questions marked , provide a well-organized solution in the answer booklet. Use mathematical statements and words to explain all of the steps of your solution. Work out some details in rough on a separate piece of paper before writing your finished solution.
5. Diagrams are *not* drawn to scale. They are intended as aids only.
6. While calculators may be used for numerical calculations, other mathematical steps must be shown and justified in your written solutions and specific marks may be allocated for these steps. For example, while your calculator might be able to find the x-intercepts of the graph of an equation like $y = x^3 - x$, you should show the algebraic steps that you used to find these numbers, rather than simply writing these numbers down.

A Note about Bubbling

Please make sure that you have correctly coded your name, date of birth and grade on the Student Information Form, and that you have answered the question about eligibility.

1. (a) There is one pair (a, b) of positive integers for which $5a + 3b = 19$. What are the values of a and b?

   (b) How many positive integers n satisfy $5 < 2^n < 2017$?

   (c) Jimmy bought 600 Euros at the rate of 1 Euro equals \$1.50. He then converted his 600 Euros back into dollars at the rate of \$1.00 equals 0.75 Euros. How many fewer dollars did Jimmy have after these two transactions than he had before these two transactions?

2. (a) What are all values of x for which $x \neq 0$ and $x \neq 1$ and $\dfrac{5}{x(x-1)} = \dfrac{1}{x} + \dfrac{1}{x-1}$?

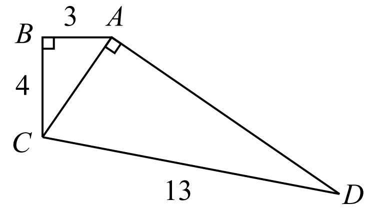
   (b) In a magic square, the numbers in each row, the numbers in each column, and the numbers on each diagonal have the same sum. In the magic square shown, what are the values of a, b and c?

0	20	a
c	4	
	-12	b

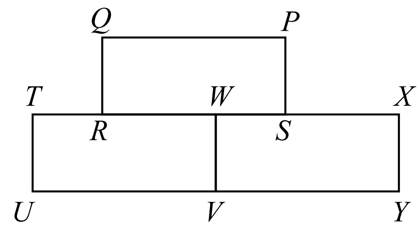
   (c) (i) For what positive integer n is $100^2 - n^2 = 9559$?

   (ii) Determine one pair (a, b) of positive integers for which $a > 1$ and $b > 1$ and $ab = 9559$.

3. (a) In the diagram, $\triangle ABC$ is right-angled at B and $\triangle ACD$ is right-angled at A. Also, $AB = 3$, $BC = 4$, and $CD = 13$. What is the area of quadrilateral $ABCD$?



(b) Three identical rectangles $PQRS$, $WTUV$ and $XWVY$ are arranged, as shown, so that RS lies along TX. The perimeter of each of the three rectangles is 21 cm. What is the perimeter of the whole shape?



(c) One of the faces of a rectangular prism has area 27 cm^2. Another face has area 32 cm^2. If the volume of the prism is 144 cm^3, determine the surface area of the prism in cm^2.

4. (a) The equations $y = a(x-2)(x+4)$ and $y = 2(x-h)^2 + k$ represent the same parabola. What are the values of a, h and k?

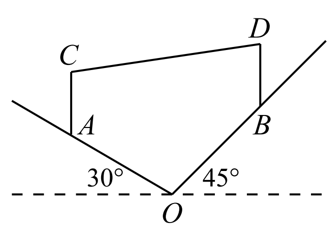
(b) In an arithmetic sequence with 5 terms, the sum of the squares of the first 3 terms equals the sum of the squares of the last 2 terms. If the first term is 5, determine all possible values of the fifth term.

(An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, $3, 5, 7, 9, 11$ is an arithmetic sequence with five terms.)

5. (a) Dan was born in a year between 1300 and 1400. Steve was born in a year between 1400 and 1500. Each was born on April 6 in a year that is a perfect square. Each lived for 110 years. In what year while they were both alive were their ages both perfect squares on April 7?

(b) Determine all values of k for which the points $A(1,2)$, $B(11,2)$ and $C(k,6)$ form a right-angled triangle.

6. (a) The diagram shows two hills that meet at O. One hill makes a 30° angle with the horizontal and the other hill makes a 45° angle with the horizontal. Points A and B are on the hills so that $OA = OB = 20$ m. Vertical poles BD and AC are connected by a straight cable CD. If $AC = 6$ m, what is the length of BD for which CD is as short as possible?



(b) If $\cos\theta = \tan\theta$, determine all possible values of $\sin\theta$, giving your answer(s) as simplified exact numbers.

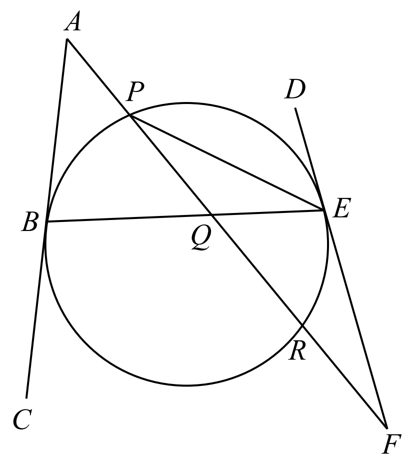
7. (a) Linh is driving at 60 km/h on a long straight highway parallel to a train track. Every 10 minutes, she is passed by a train travelling in the same direction as she is. These trains depart from the station behind her every 3 minutes and all travel at the same constant speed. What is the constant speed of the trains, in km/h?

(b) Determine all pairs (a, b) of real numbers that satisfy the following system of equations:

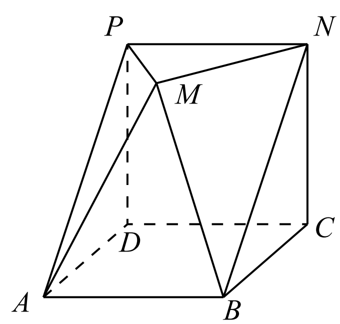
$$\begin{aligned} \sqrt{a} + \sqrt{b} &= 8 \\ \log_{10} a + \log_{10} b &= 2 \end{aligned}$$

Give your answer(s) as pairs of simplified exact numbers.

8. (a) In the diagram, line segments AC and DF are tangent to the circle at B and E, respectively. Also, AF intersects the circle at P and R, and intersects BE at Q, as shown. If $\angle CAF = 35^\circ$, $\angle DFA = 30^\circ$, and $\angle FPE = 25^\circ$, determine the measure of $\angle PEQ$.



(b) In the diagram, $ABCD$ and $PNCD$ are squares of side length 2, and $PNCD$ is perpendicular to $ABCD$. Point M is chosen on the same side of $PNCD$ as AB so that $\triangle PMN$ is parallel to $ABCD$, so that $\angle PMN = 90^\circ$, and so that $PM = MN$. Determine the volume of the convex solid $ABCDPMN$.



9. A *permutation* of a list of numbers is an ordered arrangement of the numbers in that list. For example, $3, 2, 4, 1, 6, 5$ is a permutation of $1, 2, 3, 4, 5, 6$. We can write this permutation as $a_1, a_2, a_3, a_4, a_5, a_6$, where $a_1 = 3, a_2 = 2, a_3 = 4, a_4 = 1, a_5 = 6$, and $a_6 = 5$.

(a) Determine the average value of

$$|a_1 - a_2| + |a_3 - a_4|$$

over all permutations a_1, a_2, a_3, a_4 of $1, 2, 3, 4$.

(b) Determine the average value of

$$a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + a_7$$

over all permutations $a_1, a_2, a_3, a_4, a_5, a_6, a_7$ of $1, 2, 3, 4, 5, 6, 7$.

(c) Determine the average value of

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_{197} - a_{198}| + |a_{199} - a_{200}| \qquad (*)$$

over all permutations $a_1, a_2, a_3, ..., a_{199}, a_{200}$ of $1, 2, 3, 4, \ldots, 199, 200$. (The sum labelled $(*)$ contains 100 terms of the form $|a_{2k-1} - a_{2k}|$.)

10. Consider a set S that contains $m \geq 4$ elements, each of which is a positive integer and no two of which are equal. We call S *boring* if it contains four distinct integers a, b, c, d such that $a + b = c + d$. We call S *exciting* if it is not boring. For example, $\{2, 4, 6, 8, 10\}$ is boring since $4 + 8 = 2 + 10$. Also, $\{1, 5, 10, 25, 50\}$ is exciting.

(a) Find an exciting subset of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ that contains exactly 5 elements.

(b) Prove that, if S is an exciting set of $m \geq 4$ positive integers, then S contains an integer greater than or equal to $\dfrac{m^2 - m}{4}$.

(c) Define $\text{rem}(a, b)$ to be the remainder when the positive integer a is divided by the positive integer b. For example, $\text{rem}(10, 7) = 3$, $\text{rem}(20, 5) = 0$, and $\text{rem}(3, 4) = 3$.

Let n be a positive integer with $n \geq 10$. For each positive integer k with $1 \leq k \leq n$, define $x_k = 2n \cdot \text{rem}(k^2, n) + k$. Determine, with proof, all positive integers $n \geq 10$ for which the set $\{x_1, x_2, \ldots, x_{n-1}, x_n\}$ of n integers is exciting.

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For students...

Thank you for writing the 2017 Euclid Contest! Each year, more than 235 000 students from more than 75 countries register to write the CEMC's Contests.

If you are graduating from secondary school, good luck in your future endeavours! If you will be returning to secondary school next year, encourage your teacher to register you for the 2017 Canadian Senior Mathematics Contest, which will be written in November 2017.

Visit our website `cemc.uwaterloo.ca` to find

- Free copies of past contests
- Math Circles videos and handouts that will help you learn more mathematics and prepare for future contests
- Information about careers in and applications of mathematics and computer science

For teachers...

Visit our website `cemc.uwaterloo.ca` to

- Obtain information about our 2017/2018 contests
- Look at our free online courseware for high school students
- Learn about our face-to-face workshops and our web resources
- Subscribe to our free Problem of the Week
- Investigate our online Master of Mathematics for Teachers
- Find your school's contest results