The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

cemc.uwaterloo.ca

2017 Canadian Team Mathematics Contest

Individual Problems

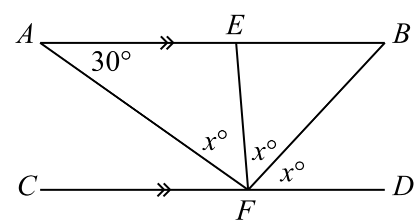
IMPORTANT NOTES:

- Calculators are allowed, with the following restriction: you may not use a device that has internet access, that can communicate with other devices, or that contains previously stored information. For example, you may not use a smartphone or a tablet.
- Express answers as simplified exact numbers except where otherwise indicated. For example, $\pi + 1$ and $1 - \sqrt{2}$ are simplified exact numbers.

PROBLEMS:

1. What is the value of x so that $\frac{8}{x} + 6 = 8$?

2. In the diagram, AB is parallel to CD. Points E and F are on AB and CD, respectively, so that $\angle FAB = 30^\circ$ and $\angle AFE = \angle EFB = \angle BFD = x^\circ$. What is the value of x?



3. The average height of Ivan, Jackie and Ken is 4% larger than the average height of Ivan and Jackie. If Ivan and Jackie are each 175 cm tall, how tall is Ken?

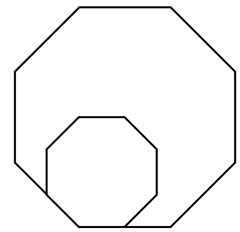
4. A positive integer n between 10 and 99, inclusive, is chosen at random. If every such integer is equally likely to be chosen, what is the probability that the sum of the digits of n is a multiple of 7?

5. A car and a minivan drive from Alphaville to Betatown. The car travels at a constant speed of 40 km/h and the minivan travels at a constant speed of 50 km/h. The minivan passes the car 10 minutes before the car arrives at Betatown. How many minutes pass between the time at which the minivan arrives in Betatown and the time at which the car arrives in Betatown?

6. Ruxandra wants to visit Singapore, Mongolia, Bhutan, Indonesia, and Japan. In how many ways can she order her trip to visit each country exactly once, with the conditions that she cannot visit Mongolia first and cannot visit Bhutan last?

7. Liesl has a bucket. Henri drills a hole in the bottom of the bucket. Before the hole was drilled, Tap A could fill the bucket in 16 minutes, tap B could fill the bucket in 12 minutes, and tap C could fill the bucket in 8 minutes. A full bucket will completely drain out through the hole in 6 minutes. Liesl starts with the empty bucket with the hole in the bottom and turns on all three taps at the same time. How many minutes will it take until the instant when the bucket is completely full?

8. In the diagram, the two regular octagons have side lengths of 1 and 2. The smaller octagon is completely contained within the larger octagon. What is the area of the region inside the larger octagon and outside the smaller octagon?



9. The parabolas with equations $y = x^2 - 2x - 3$ and $y = -x^2 + 4x + c$ intersect at points A and B. Determine the value of c so that the sum of the x-coordinate and y-coordinate of the midpoint of AB is 2017.

10. Determine the number of pairs of integers, (a, b), with $1 \leq a \leq 100$ so that the line with equation $b = ax - 4y$ passes through point $(r, 0)$, where r is a real number with $0 \leq r \leq 3$, and passes through point $(s, 4)$, where s is a real number with $2 \leq s \leq 4$.

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Team Problems

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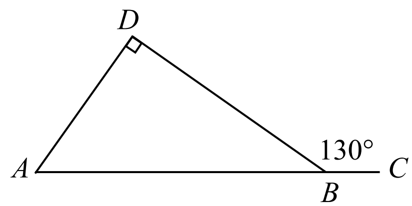
PROBLEMS:

1. In the diagram, A, B and C form a straight angle. If $\triangle ADB$ is right-angled at D and $\angle DBC = 130°$, what is the measure of $\angle DAB$?



2. Evaluate $\left[\left(\dfrac{2017+2017}{2017}\right)^{-1} + \left(\dfrac{2018}{2018+2018}\right)^{-1}\right]^{-1}$.

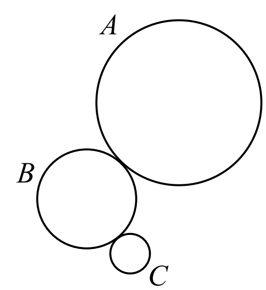
3. Bethany is told to create an expression from $2 \square 0 \square 1 \square 7$ by putting a $+$ in one box, a $-$ in another, and a $\times$ in the remaining box. There are 6 ways in which she can do this. She calculates the value of each expression and obtains a maximum value of M and a minimum value of m. What is $M - m$?

4. If n is the largest positive integer with $n^2 < 2018$ and m is the smallest positive integer with $2018 < m^2$, what is $m^2 - n^2$?

5. If N is a positive integer with $\sqrt{12} + \sqrt{108} = \sqrt{N}$, determine the value of N.

6. The ratio of the width to the height of a rectangular screen is $3 : 2$. If the length of a diagonal of the screen is 65 cm, what is the area of the screen, in cm^2?

7. In the diagram, wheel B touches wheel A and wheel C. Wheel A has radius 35 cm, wheel B has radius 20 cm, and wheel C has radius 8 cm. If wheel A rotates, it causes wheel B to rotate without slipping, which causes wheel C to rotate without slipping. When wheel A rotates through an angle of $72°$, what is the measure of the angle through which wheel C rotates?



8. Martina has a cylinder with radius 10 cm and height 70 cm that is closed at the bottom. Martina puts some solid spheres of radius 10 cm inside the cylinder and then closes the top end of the cylinder. If she puts the largest possible number of spheres in the cylinder, what is the volume, in cm^3, of the cylinder that is not taken up by the spheres?

9. Determine the value of the expression

$$1+2-3+4+5-6+7+8-9+10+11-12+\cdots+94+95-96+97+98-99\ .$$

(The expression consists of 99 terms. The operations alternate between two additions and one subtraction.)

10. Two vertical chords are drawn in a circle, dividing the circle into 3 distinct regions. Two horizontal chords are added in such a way that there are now 9 regions in the circle. A fifth chord is added that does not lie on top of one of the previous four chords. The maximum possible number of resulting regions is M and the minimum possible number of resulting regions is m. What is M^2+m^2?

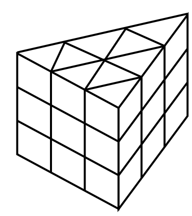
11. The product of the roots of the quadratic equation $2x^2+px-p+4=0$ is 9. What is the sum of the roots of this equation?

12. The four positive integers a,b,c,d satisfy $a<b<c<d$. When the sums of the six pairs of these integers are calculated, the six answers are all different and the four smallest sums are 6, 8, 12, and 21. What is the value of d?

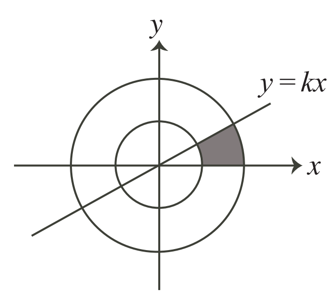
13. Determine all real values of x for which $16^x-\frac{5}{2}(2^{2x+1})+4=0$.

14. If $f(x)$ is a linear function with $f(k)=4$, $f(f(k))=7$, and $f(f(f(k)))=19$, what is the value of k?

15. A solid triangular prism is made up of 27 identical smaller solid triangular prisms, as shown. The length of every edge of each of the smaller prisms is 1. If the entire outer surface of the larger prism is painted, what fraction of the total surface area of all the smaller prisms is painted?



16. Two circles are drawn, as shown. Each is centered at the origin and the radii are 1 and 2. Determine the value of k so that the shaded region above the x-axis, below the line $y=kx$ and between the two circles has an area of 2.



17. Camp Koeller offers exactly three water activities: canoeing, swimming and fishing. None of the campers is able to do all three of the activities. In total, 15 of the campers go canoeing, 22 go swimming, 12 go fishing, and 9 do not take part in any of these activities. Determine the smallest possible number of campers at Camp Koeller.

18. Determine the area inside the circle with centre $\left(1,-\frac{\sqrt{3}}{2}\right)$ and radius 1 that lies inside the first quadrant.

19. When the polynomial $f(x)=x^4+ax^3+bx^2+cx+d$ is divided by each of $x-4$, $x-3$, $x+3$, and $x+4$, the remainder in each case is 102. Determine all values of x for which $f(x)=246$.

20. Determine the number of pairs of real numbers, (x, y), with $0 \leq x \leq \frac{\pi}{8}$ and $0 \leq y \leq \frac{\pi}{8}$ and $\cos^6(1000x) - \sin^6(1000y) = 1$.

21. Starting at midnight, Serge writes down the time after 1 minute, then 2 minutes after that, then 3 minutes after that, and so on. For example, the first four times that he writes are 12:01 a.m., 12:03 a.m., 12:06 a.m., and 12:10 a.m. He continues this process for 24 hours. What times does Serge write down that are exactly on the hour? (For example, 3:00 a.m. and 8:00 p.m. are exactly on the hour, while 11:57 p.m. is not.)

22. Figure 0 consists of a square with side length 18. For each integer $n \geq 0$, Figure $n + 1$ consists of Figure n with the addition of two new squares constructed on each of the squares that were added in Figure n. The side length of the squares added in Figure $n + 1$ is $\frac{2}{3}$ of the side length of the smallest square(s) in Figure n. Define A_n to be the area of Figure n for each integer $n \geq 0$. What is the smallest positive integer M with the property that $A_n < M$ for all integers $n \geq 0$?

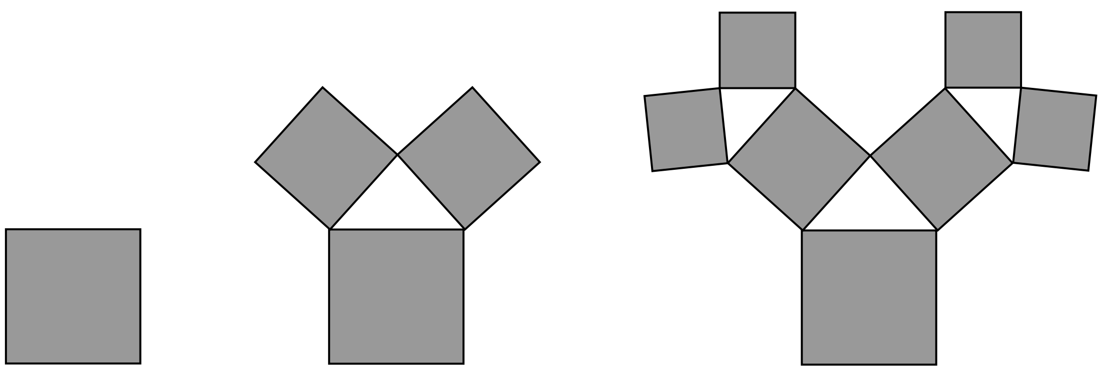
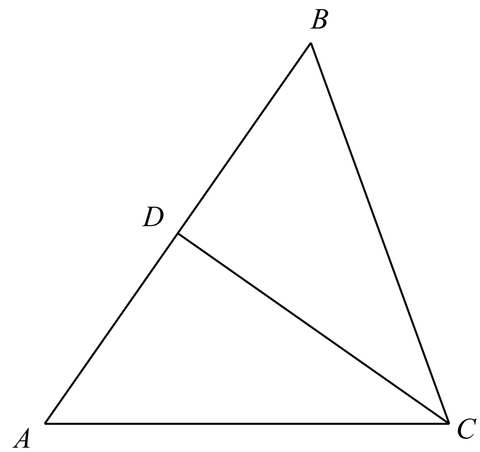


Figure 0 Figure 1 Figure 2

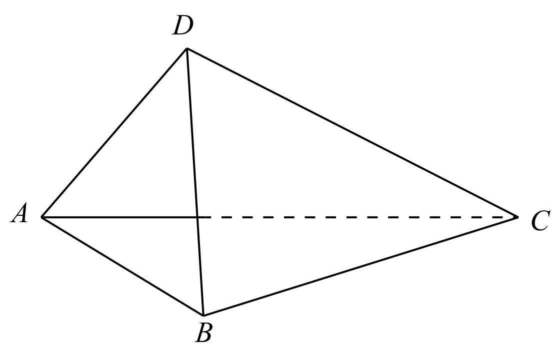
23. Brad answers 10 math problems, one after another. He answers the first problem correctly and answers the second problem incorrectly. For each of the remaining 8 problems, the probability that he answers the problem correctly equals the ratio of the number of problems that he has already answered correctly to the total number of problems that he has already answered. What is the probability that he answers exactly 5 out of the 10 problems correctly?

24. The lines with equations $x + y = 3$ and $2x - y = 0$ meet at point A. The lines with equations $x + y = 3$ and $3x - ty = 4$ meet at point B. The lines with equations $2x - y = 0$ and $3x - ty = 4$ meet at point C. Determine all values of t for which $AB = AC$.

25. A large piece of canvas is in the shape of an isosceles triangle with $AC = BC = 5$ m and $AB = 6$ m, as shown. Point D is the midpoint of AB.



The canvas is folded along median CD to create two faces of a tent ($\triangle ADC$ and $\triangle BDC$). The third face of the tent ($\triangle ABD$) is made from a separate piece of canvas. The bottom of the tent ($\triangle ABC$) is open to the ground.



What is the height of the tent (the distance from D to the base $\triangle ABC$) when the volume of the tent is as large as possible?

0 (a). Evaluate $\dfrac{9+2\times 3}{3}$.

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0 (b). Let t be TNYWR.
What is the area of a triangle with base $2t$ and height $3t-1$?

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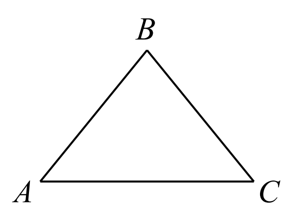
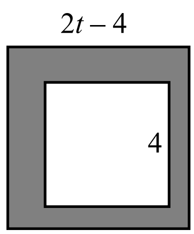
0 (c). Let t be TNYWR.
In the diagram, $\triangle ABC$ is isosceles with $AB=BC$. If $\angle BAC=t^\circ$, what is the measure of $\angle ABC$, in degrees?



1 (a). If w is a positive integer with $w^2 - 5w = 0$, what is the value of w?

1 (b). Let t be TNYWR.

In the diagram, the larger square has side length $2t - 4$ and the smaller square has side length 4. What is the area of the shaded region?



1 (c). Let t be TNYWR.

Consider the three-digit positive integers of the form $xy0$, where x and y are digits with $x \neq 0$. How many of these integers are divisible by both 11 and t?

2 (a). When the integer 300^8 is written out, it has d digits. What is the value of d?

2 (b). Let t be TNYWR.
The area of the triangle formed by the line $\sqrt{k}x + 4y = 10$, the x-axis and the y-axis is t. What is the value of k?

2 (c). Let t be TNYWR.
Justin measures the heights of three different trees: a maple, a pine and a spruce. The maple tree is 1 m taller than the pine tree and the pine tree is 4 m shorter than the spruce tree. If the ratio of the height of the maple tree to the spruce tree is t, what is the height of the spruce tree, in metres? (Write your answer in the form $\frac{a}{b}$, where a and b are positive integers with no common divisor larger than 1.)

3 (a). Suppose that $x = \sqrt{20 - 17 - 2 \times 0 - 1 + 7}$. What is the value of x?

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3 (b). Let t be TNYWR.
If the graph of $y = 2\sqrt{2t}\sqrt{x} - 2t$ passes through the point (a, a), what is the value of a?

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3 (c). Let t be TNYWR.
Suppose that

$$\frac{1}{2^{12}} + \frac{1}{2^{11}} + \frac{1}{2^{10}} + \cdots + \frac{1}{2^{t+1}} + \frac{1}{2^{t}} = \frac{n}{2^{12}}$$

(The sum on the left side consists of $13 - t$ terms.)
What is the value of n?