



The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

***2017 Canadian Senior
Mathematics Contest***

Wednesday, November 22, 2017
(in North America and South America)

Thursday, November 23, 2017
(outside of North America and South America)

Solutions

Part A

1. Calculating,

$$\frac{6}{3} \times \frac{9}{6} \times \frac{12}{9} \times \frac{15}{12} = \frac{6 \times 9 \times 12 \times 15}{3 \times 6 \times 9 \times 12} = \frac{(6 \times 9 \times 12) \times 15}{3 \times (6 \times 9 \times 12)} = \frac{15}{3} = 5$$

ANSWER: 5

- 2.
- Solution 1*

Since $\triangle AEF$ is right-angled at E , its area equals $\frac{1}{2}(AE)(EF)$.

Since $\triangle AEF$ is right-angled at E and $ABCD$ is a square, then $EF = AD = 8$ cm.

We are told that the area of $\triangle AEF$ is 30% of the area of square $ABCD$, and so the area of $\triangle AEF$ equals $0.3(8 \text{ cm})^2$.

Therefore, $\frac{1}{2}(AE)(8 \text{ cm}) = 0.3(8 \text{ cm})^2$ and so $AE = 2(0.3)(8 \text{ cm}) = 0.6(8 \text{ cm}) = 4.8$ cm.

Solution 2

Since $\triangle AEF$ is right-angled at E and $ABCD$ is a square, then EF is parallel to AD which makes $AEFD$ a rectangle.

Since AF is a diagonal of rectangle $AEFD$, then the area of rectangle $AEFD$ is twice the area of $\triangle AEF$, or 60% of the area of the square.

Since rectangle $AEFD$ and square $ABCD$ share the same height, then it must be that AE equals 60% of the length of AB .

Therefore, $AE = 0.6 \cdot AB = 0.6 \cdot 8 \text{ cm} = 4.8$ cm.

ANSWER: 4.8 cm

3. Factoring, we obtain the following equivalent equations:

$$\begin{aligned}x^4 - 3x^3 + x^2 - 3x &= 0 \\x(x^3 - 3x^2 + x - 3) &= 0 \\x[x^2(x - 3) + (x - 3)] &= 0 \\x(x - 3)[x^2 + 1] &= 0\end{aligned}$$

Therefore, $x = 0$ or $x - 3 = 0$ (which gives $x = 3$) or $x^2 + 1 = 0$ (which has no real solutions).

Therefore, $x = 0$ or $x = 3$.

ANSWER: $x = 0, 3$

- 4.
- Solution 1*

Since the two 1s are not next to each other, then the two 1s can be placed in the following pairs of positions, reading from the left: 1st and 3rd, 1st and 4th, 1st and 5th, 2nd and 4th, 2nd and 5th, 3rd and 5th.

There are 6 such pairs of positions.

Choose one of these pairs, say 1st and 3rd. This gives the number 1_1__.

There are 3 digits left to place. We place these from left to right.

There are 3 possible digits that could go in the leftmost empty position.

After this digit is placed, there are 2 possible digits that could go in the next empty position.

Finally, the remaining digit is placed in the remaining empty position.

This process works for each of the pairs of positions for the two 1s.

Therefore, there are $6 \times 3 \times 2 = 36$ such five-digit integers (6 pairs of positions for the 1s, 3 choices for the first empty position, and 2 choices for the second empty position).

Solution 2

Since there are 5 digits to arrange and 2 of them are the same, there are $\frac{5!}{2!} = \frac{120}{2} = 60$ arrangements:

To see this, replace one of the 1s with an X so that the “digits” were 1, 2, 3, 4, X . There would be 5 choices for the first digit, 4 choices for the second digit, and so on, giving $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ arrangements of the digits.

If we now replace the X with a 1, each arrangement is now counted 2 times. For example, 43 X 21 and 4312 X become the same arrangement.

Therefore, we need to divide the total 120 by $2! = 2$ since each arrangement is double-counted.

In some of these arrangements, the 1s will be next to each other and in some they will not be. We will count the arrangements with the 1s next to each other and subtract this number from the total.

If the two 1s are together, we can imagine arranging the four objects 11, 2, 3, 4.

There are $4! = 24$ such arrangements.

Thus, there are $60 - 24 = 36$ arrangements of 1, 1, 2, 3, 4 with the two 1s not next to each other.

ANSWER: 36

5. We note that the first point $(0, 0)$ attached to the integer 1 lies on the line $y = -x$. In the diagram, line segments of lengths 1, 1, 2, 2, 3, 3, 4, 4, ... are drawn starting at the origin. We guess that the points on the spiral that lie on the line $y = -x$ are those obtained after an even number of these segments are drawn. (For example, the number 3 is at the point $(1, -1)$ after 2 segments, the number 7 is at the point $(-1, 1)$ after 4 segments, the number 13 is at the point $(2, -2)$ after 6 segments, and so on.) To see why this is true, we note that, after $2k$ segments for some positive integer k , the x -coordinate of the endpoint is

$$x = 1 - 2 + 3 - \dots + (-1)^{k-1}k$$

and the y -coordinate is

$$y = -1 + 2 - 3 + \dots + (-1)^k k$$

This is because the horizontal segments have lengths 1, 2, 3, ..., k and go right, left, right, left, etc., and the vertical segments have lengths 1, 2, 3, ..., k and go down, up, down, up, etc.

Note that $-1 + 2 - 3 + \dots + (-1)^k k = -(1 - 2 + 3 - \dots + (-1)^{k-1} k)$ and so these points lie on the line $y = -x$.

We note further that no more points on the spiral can lie on the line $y = -x$ since every second point at the end of a line segment lies on the line and the segment starting at a point on the line must move to a point off of the line.

At the point after $2k$ segments, the integer written is

$$1 + (1 + 1 + 2 + 2 + 3 + 3 + \dots + k + k) = 1 + 2(1 + 2 + 3 + \dots + k) = 1 + 2\left(\frac{1}{2}k(k + 1)\right) = 1 + k + k^2$$

We want to determine the sum of the values of this expression starting with $k = 1$ and continuing until the final one that is less than or equal to 1000.

We note that the expression is increasing as k increases (as k^2 and k are both increasing), that when $k = 31$, $1 + k + k^2 = 993$, and when $k = 32$, $1 + k + k^2 = 1057$.

Therefore, we want to calculate the sum of the values of $1 + k + k^2$ from $k = 0$ to $k = 31$. (We start at $k = 0$ to include the initial integer $1 = 1 + 0 + 0^2$.)

We obtain

$$\begin{aligned} \sum_{k=0}^{31} (1 + k + k^2) &= (1 + 0 + 0^2) + (1 + 1 + 1^2) + (1 + 2 + 2^2) + (1 + 3 + 3^2) + \cdots + (1 + 31 + 31^2) \\ &= 32 \cdot 1 + (1 + 2 + 3 + \cdots + 31) + (1^2 + 2^2 + 3^2 + \cdots + 31^2) \\ &= 32 + \frac{1}{2}(31)(32) + \frac{1}{6}(31)(32)(63) \\ &\quad \text{(using the formulae given on the contest)} \\ &= 32 + 496 + 10\,416 \\ &= 10\,944 \end{aligned}$$

(The notation on the left side in the first equation is called *sigma notation* and represents the sum on the right side of the first equation.)

Therefore, the sum of the positive integers from 1 to 1000 written at points on the line $y = -x$ is 10 944.

ANSWER: 10 944

6. Since $6^2 + 8^2 = 36 + 64 = 100 = 10^2$, then the given triangle is right-angled.

We label the triangle as $\triangle ABC$ with $AB = 8$, $BC = 6$ and $AC = 10$.

We put the diagram in the xy -plane with B at the origin $(0, 0)$, A on the positive y -axis at $(0, 8)$, and C on the positive x -axis at $(6, 0)$.

Let X , Y and Z be the midpoints of sides AB , BC and AC , respectively.

Since AB , BC and AC are the diameters of the semi-circles, then X , Y and Z are the centres of the semi-circles.

Note also that since $AB = 8$, then $AX = XB = 4$, which means that the radius of the corresponding semi-circle is 4.

This also means that the coordinates of X are $(0, 4)$.

Since $BC = 6$, then $BY = YC = 3$, which means that the radius of the corresponding semi-circle is 3.

This also means that the coordinates of Y are $(3, 0)$.

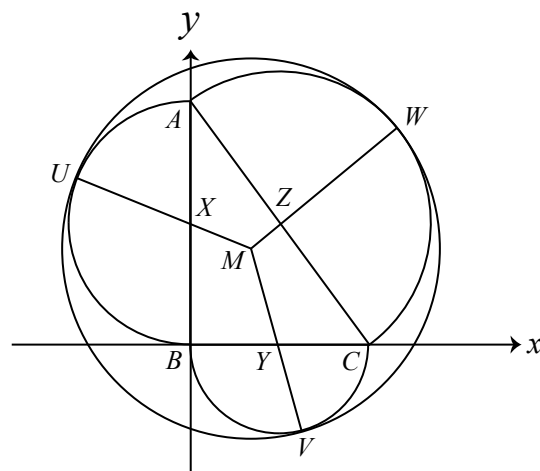
Since $AC = 10$, then $AZ = ZC = 5$, which means that the radius of the corresponding semi-circle is 5.

The coordinates of Z , the midpoint of $A(0, 8)$ and $C(6, 0)$, are $(\frac{1}{2}(0 + 6), \frac{1}{2}(8 + 0))$ or $(3, 4)$.

Let $M(s, t)$ be the centre of the large circle and r its radius.

Let the points U, V, W be the points where the semi-circles with centres X, Y and Z , respectively, touch the larger circle.

Join M to U, V and W .



It turns out that MU , MV and MW pass through X , Y and Z , respectively.

Imagine drawing a line tangent to the large circle at U .

Since the large circle and semi-circle touch at U , then this line is also tangent to the semi-circle.

Since radii are perpendicular to tangents, then XU and MU (both radii) are perpendicular to this tangent line.

Since both XU and MU are perpendicular to the tangent line, then they must lie on top of each other, and so MU passes through X .

Using a similar argument, we can see that MV and MW pass through Y and Z , respectively.

Since $MU = MV = MW = r$ (the radius of the large circle), and $XU = 4$, $YV = 3$, and $ZW = 5$ (the radii of the semi-circles), then

$$\begin{aligned} MX &= MU - XU = r - 4 \\ MY &= MV - YV = r - 3 \\ MZ &= MW - ZW = r - 5 \end{aligned}$$

The coordinates of M , X , Y , Z are $M(s, t)$, $X(0, 4)$, $Y(3, 0)$, $Z(3, 4)$.

Using the formula for the length of a line segment, we obtain the equations

$$\begin{aligned} (s - 0)^2 + (t - 4)^2 &= (r - 4)^2 \\ (s - 3)^2 + (t - 0)^2 &= (r - 3)^2 \\ (s - 3)^2 + (t - 4)^2 &= (r - 5)^2 \end{aligned}$$

Subtracting the third equation from the first, we obtain $s^2 - (s - 3)^2 = (r - 4)^2 - (r - 5)^2$ or $s^2 - s^2 + 6s - 9 = r^2 - 8r + 16 - r^2 + 10r - 25$ and so $6s - 9 = 2r - 9$ or $r = 3s$.

Subtracting the third equation from the second, we obtain $t^2 - (t - 4)^2 = (r - 3)^2 - (r - 5)^2$ or $t^2 - t^2 + 8t - 16 = r^2 - 6r + 9 - r^2 + 10r - 25$ and so $8t - 16 = 4r - 16$ or $r = 2t$.

Substituting $s = \frac{1}{3}r$ and $t = \frac{1}{2}r$ into the first equation, we obtain the following equivalent equations:

$$\begin{aligned} \left(\frac{1}{3}r\right)^2 + \left(\frac{1}{2}r - 4\right)^2 &= (r - 4)^2 \\ \frac{1}{9}r^2 + \frac{1}{4}r^2 - 4r + 16 &= r^2 - 8r + 16 \\ \frac{1}{9}r^2 + \frac{1}{4}r^2 + 4r &= r^2 \\ 4r^2 + 9r^2 + 144r &= 36r^2 \quad (\text{multiplying through by } 36) \\ 144r &= 23r^2 \\ 0 &= r(23r - 144) \end{aligned}$$

Therefore, $r = 0$ (which is impossible) or $r = \frac{144}{23}$, which is the desired answer.

ANSWER: $\frac{144}{23}$

Part B

1. (a) Factoring the left side of $x^2 + 2x - 8 = 0$, we obtain $(x + 4)(x - 2) = 0$.
Therefore, the roots of $x^2 + 2x - 8 = 0$ are $x = -4$ and $x = 2$.
- (b) Since the parabola with equation $y = x^2 + bx + c$ passes through the points $(1, 2)$ and $(2, 0)$, then the coordinates of these points satisfy the equation of the parabola.
Therefore, $2 = 1^2 + b \cdot 1 + c$ (which simplifies to $b + c = 1$) and $0 = 2^2 + 2b + c$ (which simplifies to $2b + c = -4$).
Subtracting the first of these equations from the second, we obtain $(2b + c) - (b + c) = -4 - 1$ or $b = -5$.
Since $b = -5$ and $b + c = 1$, then $c = 1 - b = 1 - (-5) = 6$.
Therefore, $b = -5$ and $c = 6$.
(We can check that the points $(1, 2)$ and $(2, 0)$ lie on the parabola with equation $y = x^2 - 5x + 6$.)
- (c) Since the point $(0, 2)$ lies on the parabola with equation $y = a(x - 1)^2 + \frac{8}{3}$, then $2 = a(-1)^2 + \frac{8}{3}$ or $2 = a + \frac{8}{3}$ and so $a = 2 - \frac{8}{3} = -\frac{2}{3}$.
Thus, the parabola has equation $y = -\frac{2}{3}(x - 1)^2 + \frac{8}{3}$.
We want to determine the value of $d > 0$ for which the point $(d, 0)$ lies on this parabola.
Substituting, we obtain the following equivalent equations:

$$\begin{aligned} 0 &= -\frac{2}{3}(d - 1)^2 + \frac{8}{3} \\ \frac{2}{3}(d - 1)^2 &= \frac{8}{3} \\ (d - 1)^2 &= 4 \\ d - 1 &= \pm 2 \end{aligned}$$

Therefore, $d = -2 + 1 = -1$ or $d = 2 + 1 = 3$.
Since $d > 0$, then $d = 3$.

2. (a) The following table shows how Joe can win the game in three turns:

| | | | | | | |
|---|---|---|---|---|---|---------------|
| R | R | R | R | R | R | Start |
| G | G | G | G | R | R | After 1 turn |
| G | R | R | R | G | R | After 2 turns |
| G | G | G | G | G | G | After 3 turns |

On Joe's first turn, he turns over the first 4 cards.
On Joe's second turn, he turns over cards 2 through 5.
On Joe's third turn, he turns over the four red cards.

- (b) There are many sequences of moves in which Joe can win.

Suppose that Joe takes 9 turns. On turn 1, he turns over cards 1, 2, 3, 4, 5. On turn 2, he turns over cards 2, 3, 4, 5, 6.

He continues in this way so that, on each turn, he turns over five consecutive cards starting with the t th card on turn t , with the understanding that card 1 comes after card 9. This means, for example, that on turn 7, Joe turns over cards 7, 8, 9, 1, 2.

In this way, each of the 9 cards is turned over 5 times (once as each of the 1st, 2nd, 3rd, 4th, 5th card in the sequence).

Since each card is turned over an odd number of times, its final colour is the opposite of the starting colour, and so it is green.

We demonstrate this in the following chart:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---------------|
| R | R | R | R | R | R | R | R | R | Start |
| G | G | G | G | G | R | R | R | R | After 1 turn |
| G | R | R | R | R | G | R | R | R | After 2 turns |
| G | R | G | G | G | R | G | R | R | After 3 turns |
| G | R | G | R | R | G | R | G | R | After 4 turns |
| G | R | G | R | G | R | G | R | G | After 5 turns |
| R | R | G | R | G | G | R | G | R | After 6 turns |
| G | G | G | R | G | G | G | R | G | After 7 turns |
| R | R | R | R | G | G | G | G | R | After 8 turns |
| G | G | G | G | G | G | G | G | G | After 9 turns |

Joe can actually finish in as few as three turns:

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---------------|
| R | R | R | R | R | R | R | R | R | Start |
| G | G | G | G | G | R | R | R | R | After 1 turn |
| G | G | R | R | R | G | G | R | R | After 2 turns |
| G | G | G | G | G | G | G | G | G | After 3 turns |

- (c) Suppose that $n = 2017$. This means that Joe has 2017 cards.

We show that Joe can win the game when k is odd and cannot win the game when k is even.

Suppose that k is odd.

Suppose that Joe takes 2017 turns.

For each $t = 1, 2, 3, \dots, 2016, 2017$, Joe turns over the k cards starting at card t , and with the understanding that card 1 comes after card 2017.

In this way, each of the 2017 cards is turned over k times, once for each “position” in a sequence of k consecutive cards.

Since k is odd, then the colour of each of the 2017 cards is reversed at the end, and so each is green.

In this way, Joe wins the game when k is odd.

Suppose that k is even.

For Joe to win the game, each of the 2017 cards must be turned over an odd number of times in order to reverse its colour.

This means that the total number of card flips is odd, since this total is the sum of 2017 odd integers (the number of flips for each of the 2017 cards).

For any positive integer t , after t turns, Joe has flipped a total of tk cards (k on each of t turns).

Since k is even, then tk is even.

Therefore, after any number of turns, the total number of card flips is always even and so cannot be the odd number of flips necessary to reverse the colour of all of the cards.

Therefore, when k is even, Joe cannot win the game.

In summary, when $n = 2017$, Joe can win the game for all odd k with $1 \leq k < 2017$ and cannot win the game for all even k with $1 \leq k < 2017$.

3. (a) Continuing to substitute values of n back into $f(n)$, we obtain:

$$\begin{aligned} n = 2 : \quad f(f(2)) &= f(2) + 3 \cdot 2 \\ f(5) &= 5 + 6 = 11 \end{aligned}$$

$$\begin{aligned} n = 5 : \quad f(f(5)) &= f(5) + 3 \cdot 5 \\ f(11) &= 11 + 15 = 26 \end{aligned}$$

$$\begin{aligned} n = 11 : \quad f(f(11)) &= f(11) + 3 \cdot 11 \\ f(26) &= 26 + 33 = 59 \end{aligned}$$

Therefore, $f(26) = 59$.

- (b) We show that there is no such function g by using the given information to derive two different outputs for a single input, which will give a contradiction of the important property of a function that there is only one output for a given input.

At each step, we use values of m and n for which we already know the values of $g(m)$ and $g(n)$, which limits our options at each step.

Assume that there exists a function g with $g(1) = 2$ and $g(g(n) + m) = n + g(m)$.

- If $n = 1$ and $m = 1$, then $g(g(1) + 1) = 1 + g(1)$ or $g(2 + 1) = 1 + 2$ and so $g(3) = 3$.
- If $n = 1$ and $m = 3$, then $g(g(1) + 3) = 1 + g(3)$ or $g(2 + 3) = 1 + 3$ and so $g(5) = 4$.
- If $n = 3$ and $m = 1$, then $g(g(3) + 1) = 3 + g(1)$ or $g(3 + 1) = 3 + 2$ and so $g(4) = 5$.
- If $n = 3$ and $m = 2$, then $g(g(3) + 2) = 3 + g(2)$ or $g(3 + 2) = 3 + g(2)$.
Since $g(5) = 4$, then $4 = 3 + g(2)$ or $g(2) = 1$.
- If $n = 2$ and $m = 4$, then $g(g(2) + 4) = 2 + g(4)$ or $g(1 + 4) = 2 + 5$ and so $g(5) = 7$.

Since $g(5) = 4$ and $g(5) = 7$, we have a contradiction.

Therefore, there does not exist such a function g .

(c) *Solution 1*

First, we show that the function $h(n) = n + 1$ for each positive integer n satisfies the conditions.

We note that this function satisfies the first two conditions.

Thus, if $h(n) = n + 1$ for each positive integer n ,

$$h(h(n) + m) = h(n + 1 + m) = (n + 1 + m) + 1 = n + m + 2$$

and

$$1 + n + h(m) = 1 + n + m + 1 = n + m + 2$$

and so the third condition is satisfied.

Next, we show that there is no other such function.

Suppose that the function h satisfies the three conditions and that $h(1) = k$.

Step 1: Show that $h(a + rk) = h(a) + 2r$ for each integer $r \geq 1$

Using the functional equation with $n = 1$ and $m = a$, we obtain $h(h(1) + a) = 1 + 1 + h(a)$ or $h(a + k) = h(a) + 2$.

Using the functional equation with $n = 1$ and $m = a + k$, we obtain

$$h(h(1) + a + k) = 1 + 1 + h(a + k)$$

or $h(a + 2k) = 2 + h(a) + 2 = h(a) + 4$.

Using the functional equation with $n = 1$ and $m = a + 2k$, we obtain

$$h(h(1) + a + 2k) = 1 + 1 + h(a + 2k)$$

or $h(a + 3k) = 2 + h(a) + 4 = h(a) + 6$.

Assume that $h(a + (r - 1)k) = h(a) + 2(r - 1)$ for some integer $r \geq 2$.

Using the functional equation with $n = 1$ and $m = a + (r - 1)k$, we obtain

$$h(h(1) + a + (r - 1)k) = 1 + 1 + h(a + (r - 1)k)$$

or $h(a + rk) = 2 + h(a) + 2(r - 1) = h(a) + 2r$.

Step 2: Show that if $h(b) = h(c)$, then $b = c$

In other words, we show that the function h is one-to-one.

Suppose that $h(b) = h(c)$.

Then for each positive integer d , $h(h(b) + d) = 1 + b + h(d)$ and $h(h(c) + d) = 1 + c + h(d)$. Since $h(b) = h(c)$, then $h(h(b) + d) = h(h(c) + d)$ and so $1 + b + h(d) = 1 + c + h(d)$ or $b = c$.

Step 3: Show that $k \leq 2$

From Step 1, $h(a + rk) = h(a) + 2r$ for all positive integers a and r .

Suppose that $k \geq 3$.

When $a = 1$, we obtain $h(1 + rk) = h(1) + 2r$ for each positive integer r .

The values of h here form an infinite set of positive integers starting with $h(1) + 2$ which are spaced 2 apart.

When $a = 2$, we obtain $h(2 + rk) = h(2) + 2r$ for each positive integer r .

The values of h here form an infinite set of positive integers starting with $h(2) + 2$ which are spaced 2 apart.

When $a = 3$, we obtain $h(3 + rk) = h(3) + 2r$ for each positive integer r .

The values of h here form an infinite set of positive integers starting with $h(3) + 2$ which are spaced 2 apart.

Since $k \geq 3$, then the inputs in these three sets are all distinct, since those in the first family give remainder 1 when divided by k , those in the second family give remainder 2 when divided by k , and those in the third family give remainder 3 (or possibly 0 if $k = 3$). But the three sets of outputs $h(1) + 2r$, $h(2) + 2r$, $h(3) + 2r$ must overlap eventually, as there can be only two disjoint infinite sets with common difference 2.

This is a contradiction, so we cannot have $k \geq 3$, which means that $k \leq 2$.

Step 4: Show that $k = 2$

We know that $k = 1$ or $k = 2$.

If $h(1) = k = 1$, the equation $h(1 + rk) = h(1) + 2r$ becomes $h(1 + r) = 1 + 2r$.

Substituting $n = 1 + r$ (or $r = n - 1$) gives $h(n) = 1 + 2(n - 1) = 2n - 1$.

In this case,

$$h(h(n) + m) = h(2n - 1 + m) = 2(2n - 1 + m) - 1 = 4n + 2m - 3$$

and

$$1 + n + h(m) = 1 + n + 2m - 1 = n + 2m$$

These two expressions are not always equal for all n and m (for example, when $n = m = 2$). Therefore, we cannot have $k = 1$. Thus, $k = 2$.

Step 5: $h(n) = n + 1$ when n is odd

From $h(1 + rk) = h(1) + 2r$ and $k = 2$, we obtain $h(1 + 2r) = 2 + 2r = (2r + 1) + 1$ and so $h(n) = n + 1$ when n is odd.

Step 6: $h(n) = n + 1$ when n is even

From $h(2 + rk) = h(2) + 2r$, we obtain $h(2 + 2r) = h(2) + 2r$.

If we can show that $h(2) = 3$, then we will obtain $h(2 + 2r) = 3 + 2r$ as required.

Since $h(x)$ is even when x is odd and since h is one-to-one, then $h(x)$ must be odd when x is even.

In particular, $h(2)$ is odd.

From $h(h(n) + m) = 1 + n + h(m)$ with $n = 2$ and $m = 1$, we get $h(h(2) + 1) = 1 + 2 + h(1)$ and so $h(h(2) + 1) = 5$.

When n is even, the values of $h(n)$ form an increasing sequence of odd positive integers. This is because $h(2)$ is odd and so $h(2) + 2r$ is odd and increasing as r increases, and so $h(2 + 2r) = h(2) + 2r$ is odd and increasing as r increases.

Therefore, $h(h(2) + 1) = 5$ has to be the third smallest, or second smallest, or smallest value of $h(n)$ when n is even.

Since $h(2 + 2r) = h(2) + 2r$, then the values of $h(x)$ when x is even increase when x increases by 2.

This means that we must have $h(6) = 5$ or $h(4) = 5$ or $h(2) = 5$.

If $h(6) = 5$, then $h(h(2) + 1) = 5$ gives $h(2) + 1 = 6$ or $h(2) = 5$, which cannot be the case, since h is one-to-one.

If $h(2) = 5$, then $h(h(2) + 1) = 5$ gives $h(2) + 1 = h(2)$ which cannot be the case.

Therefore, $h(4) = 5$ and so $h(2 + 2) = h(2) + 2$ which gives $h(2) = 5 - 2 = 3$.

We have shown that the function $h(n) = n + 1$ satisfies the given condition and that no other function does, as required.

Solution 2

Let h be a function that satisfies the given properties, and consider the infinite set of lattice points of the form $(m, h(m))$, for $m = 1, 2, 3, \dots$

Let a be a positive integer.

When $n = a$ and $m = h(a)$, the functional equation is $h(h(a) + h(a)) = 1 + a + h(h(a))$ or $h(2h(a)) = 1 + a + h(h(a))$.

Thus, the point $(2h(a), 1 + a + h(h(a)))$ is on the graph of $y = h(x)$.

When $n = a$ and $m = 2h(a)$, the functional equation becomes

$$h(h(a) + 2h(a)) = 1 + a + h(2h(a))$$

or $h(3h(a)) = 1 + a + 1 + a + h(h(a)) = 2 + 2a + h(h(a))$.

Thus, the point $(3h(a), 2 + 2a + h(h(a)))$ is on the graph of $y = h(x)$.

Suppose that r is a positive integer and assume that $h(rh(a)) = (r-1) + (r-1)a + h(h(a))$.

When $n = a$ and $m = rh(a)$, the functional equation becomes

$$h(h(a) + rh(a)) = 1 + a + h(rh(a)) = 1 + a + (r-1) + (r-1)a + h(h(a)) = r + ra + h(h(a))$$

or $h((r+1)h(a)) = r + ra + h(h(a))$.

Thus, if the point $(rh(a), (r-1) + (r-1)a + h(h(a)))$ is on the graph of $y = h(x)$, then the point $((r+1)h(a), r + ra + h(h(a)))$ is on the graph of $y = h(x)$.

This shows that the infinite set of points of the form $(sh(a), (s-1) + (s-1)a + h(h(a)))$ for integers $s \geq 2$ all lie on this graph.

Note that to get from one point in this set to the next, we move $h(a)$ units to the right and $a + 1$ units up.

In other words, this infinite set of points on the graph forms a line with slope $\frac{a+1}{h(a)}$.

Call this line L_a .

Suppose that b is a positive integer with $b \neq a$.

A similar argument shows that the set of points $(th(b), (t-1) + (t-1)b + h(h(b)))$ for integers $t \geq 2$ all lie on this graph, which thus form a line L_b with slope $\frac{b+1}{h(b)}$.

Now consider the points $(n, h(n))$ on the graph of $y = h(x)$ where n is a positive integer that is a multiple of $h(a)$ and is also a multiple of $h(b)$.

There are infinitely many such points (for example, $n = wh(a)h(b)$ for positive integers w).

These points lie on both L_a and L_b , and so the slopes of these two lines must be equal.

But this is true for every possible pair of distinct positive integers a and b .

Therefore, the quantity $\frac{a+1}{h(a)}$ must be constant for all positive integers a . (Another way to see this is to let $a = 1$ and let b range through all positive integers greater than 1.)

In other words, $\frac{a+1}{h(a)} = \frac{1}{c}$ for some constant c or $h(a) = ca + c$ for all positive integers a .

If $a = 1$, then $h(1) = c + c = 2c$.

Setting $n = m = 1$ in the equation $h(h(n) + m) = 1 + n + h(m)$, we obtain $h(h(1) + 1) = 1 + 1 + h(1)$ which gives $h(2c + 1) = 2 + 2c$.

Since $h(a) = ca + c$, then setting $a = 2c + 1$ gives $c(2c + 1) + c = 2 + 2c$ or $2c^2 + 2c = 2 + 2c$ and so $c^2 = 1$.

Since $c > 0$, then $c = 1$ and so $h(n) = n + 1$.

(Solution 1 shows us that the function $h(n) = n + 1$ does satisfy the requirements.)