The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2017 Canadian Intermediate Mathematics Contest

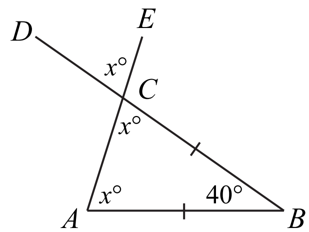
Wednesday, November 22, 2017
(in North America and South America)

Thursday, November 23, 2017
(outside of North America and South America)

Solutions

Part A

1. *Solution 1*
   Since opposite angles are equal, $\angle ACB = \angle DCE = x^\circ$.
   In $\triangle ABC$, $AB = BC$, and so $\angle CAB = \angle ACB = x^\circ$.



   Therefore,

$$\begin{aligned}\angle ACB + \angle CAB + \angle ABC &= 180^\circ\\ x^\circ + x^\circ + 40^\circ &= 180^\circ\\ 2x &= 140\end{aligned}$$

   and so $x = 70$.

   Solution 2
   In $\triangle ABC$, $AB = BC$, and so $\angle CAB = \angle ACB$.
   Since the sum of the angles in $\triangle ABC$ is 180°, then

$$\begin{aligned}\angle ACB + \angle CAB + \angle ABC &= 180^\circ\\ 2\angle ACB + 40^\circ &= 180^\circ\\ 2\angle ACB &= 140^\circ\\ \angle ACB &= 70^\circ\end{aligned}$$

   Since $\angle DCE$ and $\angle ACB$ are opposite angles, then $x^\circ = 70^\circ$ and so $x = 70$.

   ANSWER: $x = 70$

2. We list the twelve integers from smallest to largest:

$$1277, 1727, 1772, 2177, 2717, 2771, 7127, 7172, 7217, 7271, 7712, 7721$$

   The sum of the 6th and 7th integers in the list is $2771 + 7127 = 9898$.

   ANSWER: 9898

3. *Solution 1*
   Each of the three column sums represents the total of the three entries in that column.
   Therefore, the sum of the three column sums is equal to the sum of the nine entries in the table.
   Thus, the sum of the nine entries in the table is $22 + 12 + 20 = 54$.
   Similarly, the sum of the three row sums must equal the sum of the nine entries in the table.
   Thus, $x + 20 + 15 = 54$ or $x = 19$.

Solution 2
From the second column, $\triangle + \triangle + \triangle = 12$ or $3 \times \triangle = 12$ and so $\triangle = 4$.
From the second row, $\square + \triangle + \square = 20$.
Since $\triangle = 4$, then $2 \times \square = 16$ and so $\square = 8$.
From the first column, $\heartsuit + \square + \heartsuit = 22$.
Since $\square = 8$, then $2 \times \heartsuit = 14$ and so $\heartsuit = 7$.
From the first row, $x = \heartsuit + \triangle + \square = 7 + 4 + 8 = 19$.

ANSWER: $x = 19$

4. Since the top of the cookie is a circle with radius 3 cm, its area is $\pi(3\text{ cm})^2 = 9\pi\text{ cm}^2$.
Since a chocolate chip has radius 0.3 cm, its area is $\pi(0.3\text{ cm})^2 = 0.09\pi\text{ cm}^2$.
We are told that k chocolate chips cover $\frac{1}{4}$ of the area of the top of the cookie.
Therefore, $k \times 0.09\pi\text{ cm}^2 = \frac{1}{4} \times 9\pi\text{ cm}^2$ and so $0.09k = \frac{1}{4} \times 9 = 2.25$ or $k = \frac{2.25}{0.09} = 25$.

ANSWER: $k = 25$

5. Since $\dfrac{31}{72} = \dfrac{a}{8} + \dfrac{b}{9} - c$, then multiplying both sides by 72 gives $31 = 9a + 8b - 72c$.
Rearranging, we obtain $72c - 9a = 8b - 31$.
Since a and c are positive integers, then $72c - 9a$ is an integer.
Since $72c - 9a = 9(8c - a)$, then $72c - 9a$ must be a multiple of 9.
Since $72c - 9a$ is a multiple of 9, then $8b - 31 = 72c - 9a$ must also be a multiple of 9.
When $b = 1, 2, 3, 4$, we obtain $8b - 31 = -23, -15, -7, 1$, none of which is a multiple of 9.
When $b = 5$, we obtain $8b - 31 = 9$, which is a multiple of 9.
Note further that if $b = 5$, then the positive integers $c = 1$ and $a = 7$ gives $72c - 9a = 9 = 8b - 31$.
(There are other values of c and a that will give $72c - 9a = 9$, including $c = 2$ and $a = 15$.)
Thus, when $b = 1, 2, 3, 4$, there are no solutions, and when $b = 5$, there is at least one solution.
Therefore, the smallest possible value of b is 5.

ANSWER: 5

6. In the grid, there are 5 empty cells.
Ignoring the restrictions on placement, there are 3 choices of letter that can mark each square. This means that there are $3^5 = 243$ possible grids.
To count the number of ways in which the grid can be completed so that it includes at least one pair of squares side-by-side in the same row or same column that contain the same letter, we count the number of ways in which the grid can be completed so that no pair of side-by-side squares contains the same letter, and subtract this total from 243.
We focus on the top left and bottom middle squares. Each of these must be S or T.
We consider the four possible cases.

Case 1:

S	R	
	S	

The bottom left square cannot be S, but can be R or T (2 choices).
The top right square cannot be R, and so can be T or S.
If the top right square is T, then the bottom right square cannot be S or T and so must be R.
If the top right square is S, then the bottom right square can be R or T (2 choices).
This means that there are 2 possibilities for the left side of the grid and 3 for the right side of the grid. These sets of possibilities are independent and so there are $2 \times 3 = 6$ configurations in this case.

These are

S	R	T
R	S	R

,

S	R	S
R	S	R

,

S	R	S
R	S	T

,

S	R	T
T	S	R

,

S	R	S
T	S	R

,

S	R	S
T	S	T

.

Case 2:

T	R	
	T	

As in Case 1, there are 6 configurations. These can be obtained from those in Case 1 by changing the T's to S's and the S's to T's in the list of possibilites in Case 1.

Case 3:

S	R	
	T	

Here, the bottom left cell must be R.
As in Case 1, there are 3 possible ways to fill the right side.

This gives 3 configurations:

S	R	T
R	T	R

,

S	R	T
R	T	S

,

S	R	S
R	T	R

.

Case 4:

T	R	
	S	

As in Case 3, there are 3 configurations. These can be obtained from those in Case 3 by changing the T's to S's and the S's to T's in the list of possibilities from Case 3.

In total, there are thus $6+6+3+3 = 18$ configurations for which there are no two side-by-side cells that contain the same letter.
This means that there are $243 - 18 = 225$ configurations in which there are two side-by-side cells that contain the same letter.

ANSWER: 225

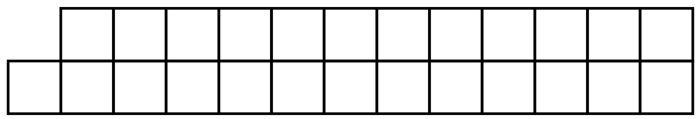
Part B

1. (a) Since Figure 1 is formed by 3 squares and each subsequent Figure has 2 additional squares, then Figure 8 is formed by $3 + 7 \times 2 = 17$ squares.

(b) *Solution 1*

Figure 1 is formed by 3 squares, and each subsequent Figure includes one extra square in each row.

Thus, Figure 12 consists of a top row of 12 squares and a bottom row of 13 squares.



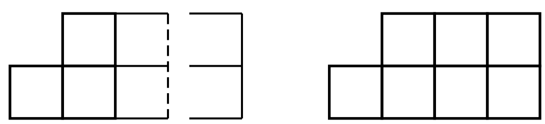
The perimeter of Figure 12 equals 12 cm (top sides of top row) plus 2 cm (right side of figure) plus 13 cm (bottom sides of bottom row) plus 3 cm ("up, right, up" on left side), or 30 cm.

Solution 2

Figure 1 is formed by 3 squares, and each subsequent Figure includes one extra square in each row.

The perimeter of Figure 1 is 8 cm.

When moving from one Figure to the next, the outer edges change in the following way: a length of 1 cm is added to each of the top and bottom rows, and the right side (length 2 cm) of the original Figure is replaced with the right side (length 2 cm) of the new Figure.



Therefore, when moving from one Figure to the next, the perimeter increases by 2 cm.

Since the perimeter of Figure 1 is 8 cm, then the perimeter of Figure 12 is equal to $8 \text{ cm} + 11 \times 2 \text{ cm} = 30 \text{ cm}$.

(c) *Solution 1*

As in Solution 2 to (b), we notice that moving from one figure to the next increases the perimeter by 2 cm.

Since Figure 1 has perimeter 8 cm and Figure C has perimeter 38 cm, then Figure C must be $\dfrac{30 \text{ cm}}{2 \text{ cm}} = 15$ figures further along in the sequence from Figure 1.

In other words, $C = 16$.

Solution 2

As in Solution 2 to (b), we notice that moving from one figure to the next increases the perimeter by 2 cm.

Since Figure C is $C-1$ Figures further along in the sequence from Figure 1, the perimeter of Figure C is

$$8 \text{ cm} + (C-1) \times 2 \text{ cm} = 8 \text{ cm} + (2C-2) \text{ cm} = (2C+6) \text{ cm}$$

For this perimeter to equal 38 cm, we must have $2C + 6 = 38$ or $2C = 32$ and so $C = 16$.

(d) Using a similar argument to that in Solution 2 to (c), we see that the perimeter of Figure D is $(2D+6)$ cm.
Using the formula from Solution 2 to (c) with $C = 29$, we see that the perimeter of Figure 29 is $(2 \times 29 + 6)$ cm $= 64$ cm.
From the given information, we want $\dfrac{64 \text{ cm}}{(2D+6) \text{ cm}} = \dfrac{4}{11}$ or $\dfrac{64}{2D+6} = \dfrac{4}{11}$.
Since $64 = 4 \times 16$, then $\dfrac{4}{11} = \dfrac{4 \times 16}{11 \times 16} = \dfrac{64}{176}$.
Therefore, $\dfrac{64}{2D+6} = \dfrac{4}{11} = \dfrac{64}{176}$ implies $2D + 6 = 176$ and so $2D = 170$ or $D = 85$.

2. (a) Since $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ and $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$, then $\dfrac{7!}{5!} = \dfrac{5040}{120} = 42$.
Alternatively, we could note that

$$\frac{7!}{5!} = \frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{5 \times 4 \times 3 \times 2 \times 1} = \frac{7 \times 6 \times (5 \times 4 \times 3 \times 2 \times 1)}{(5 \times 4 \times 3 \times 2 \times 1)} = 7 \times 6 = 42$$

(b) Since $98! = 98 \times 97 \times 96 \times \cdots \times 3 \times 2 \times 1$ and $9900 = 100 \times 99$, then

$$98! \times 9900 = (98 \times 97 \times 96 \times \cdots \times 3 \times 2 \times 1) \times (100 \times 99) = 100 \times 99 \times 98 \times 97 \times 96 \times \cdots \times 3 \times 2 \times 1$$

By definition, this is $100!$ and so $n = 100$.

(c) Since $(m+2)!$ is the product of all positive integers from 1 to $m+2$, then

$$(m+2)! = (m+2) \times (m+1) \times m \times (m-1) \times \cdots \times 3 \times 2 \times 1 = (m+2) \times (m+1) \times m!$$

Therefore, $\dfrac{(m+2)!}{m!} = \dfrac{(m+2) \times (m+1) \times m!}{m!} = (m+2) \times (m+1)$.
Since $\dfrac{(m+2)!}{m!} = 40\,200$, then $(m+2)(m+1) = 40\,200$.
Since $40\,200 = 100 \times 402 = 100 \times 2 \times 201 = 200 \times 201$, then when $m + 1 = 200$, we have $(m+2)(m+1) = 201 \times 200 = 40\,200$.
Therefore, $m + 1 = 200$ and so $m = 199$.

(d) Since $(q+2)! - (q+1)! = (q!) \times r$, then

$$r = \frac{(q+2)! - (q+1)!}{q!} = \frac{(q+2)!}{q!} - \frac{(q+1)!}{q!}$$

Modelling our approach in (c), $\dfrac{(q+2)!}{q!} = (q+2)(q+1)$ and $\dfrac{(q+1)!}{q!} = \dfrac{(q+1) \times q!}{q!} = q+1$.
Thus, $r = (q+2)(q+1) - (q+1)$.
Since q is an integer, then r is an integer.
Furthermore, $r = (q+1)[(q+2)-1] = (q+1)[q+1] = (q+1)^2$, which is a perfect square, as required.

3. Throughout this solution, we use the appreviation BPI to represent "balanced positive integer". In addition, we call a BPI of the form $abc\,def$ an abc-BPI.

(a) The integer $3b8\,d5f$ is a BPI exactly when $3 \times b \times 8 = d \times 5 \times f$ or $24 \times b = 5 \times d \times f$.
Since the right side is an integer that is a multiple of 5, then $24 \times b$ must be a multiple of 5.
Since b is a non-zero digit, then the only possible value of b that makes $24 \times b$ a multiple of 5 is $b = 5$.
Therefore, $24 \times 5 = 5 \times d \times f$ which is equivalent to $d \times f = 24$.
Since d and f are non-zero digits, then d and f must be 3 and 8, or 4 and 6, or 6 and 4, or 8 and 3.
Therefore, there are four BPIs of the form $3b8\,d5f$:

$$358\,358, 358\,456, 358\,654, 358\,853$$

(b) If the digits $4, b, c$ are all distinct, then there are at least 6 $4bc$-BPIs, since the last three digits could be any of $4bc, 4cb, b4c, bc4, c4b, cb4$ (and possibly others).
Since we are looking for exactly three $4bc$-BPIs, then the digits $4, b, c$ cannot all be distinct.
Thus, at least two of $4, b, c$ must be equal.
Consider $b = c = 7$.
Here, $4 \times b \times c = 4 \times 7 \times 7 = 196$ and so for $4bc\,def$ to be a BPI, we need $d \times e \times f = 196$.
Since 196 has two factors of 7, then $d \times e \times f$ has two factors of 7.
Since each of d, e, f is a digit, then two of d, e, f must equal 7 (7 is the only digit divisible by 7).
This means that the remaining digit equals $\dfrac{196}{7 \times 7} = 4$ and so d, e, f equal $4, 7, 7$ in some order.
There are three arrangements of the digits $4, 7, 7$, namely $477, 747, 774$.
Therefore, if $4bc$ is 477, then there are exactly three BPIs of the form $4bc\,def$.
We note that $4bc$ could also equal 455 or 488.

(c) Consider a fixed three digit integer abc.
If $a \times b \times c = d \times e \times f$ and d, e, f are all different, then there are exactly 6 abc-BPIs with final digits d, e, f in some order: $abc\,def, abc\,dfe, abc\,edf, abc\,efd, abc\,fde, abc\,fed$.
These come from the 6 different arrangements of d, e, f.
If $a \times b \times c = d \times e \times e$ and d and e are different, then there are exactly 3 abc-BPIs with final digits d, e, e in some order $abc\,dee, abc\,ede, abc\,eed$.
If $a \times b \times c = d \times d \times d$, then there is exactly 1 abc-BPI with final digits d, d, d in some order: $abc\,ddd$.
This means that the abc-BPIs come in groupings of 6, 3 or 1, with one or more such groupings associated to a fixed abc.
Next, we note that for a fixed abc, there cannot be two BPIs of the form $abc\,ddd$ and $abc\,eee$ with d and e different:

> If this were possible, then $a \times b \times c = d \times d \times d$ and $a \times b \times c = e \times e \times e$, which would give $d \times d \times d = e \times e \times e$ or $d^3 = e^3$.
> Since d and e are positive digits, then $d = e$ which means that $abc\,ddd$ and $abc\,eee$ are the same.

Since there can be at most 1 abc-BPI of the form $abc\,ddd$ and all other abc-BPIs come in groupings of 3 or 6, then the total number of abc-BPIs is either a multiple of 3 or is 1 more than a multiple of 3.

We note as well that if abc and ABC satisfy $a \times b \times c = A \times B \times C$, then there are exactly the same number of abc-BPIs and ABC-BPIs, because it is the product of the first three digits that determines the possible last three digits, not the exact digits themselves. In the analysis below, this means that we can look at a specific abc attached to a given def and not be concerned that a different ABC will give a different total count.
We are now ready to tackle the given problem by looking at various values of k.

Since the number of abc-BPIs is either a multiple of 3 or 1 more than a multiple of 3 for any abc, then for $k = 5$ and $k = 8$, there does not exist an abc with exactly k abc-BPIs.
Let us look next at $k = 4, 7, 10$, each of which is 1 more than a multiple of 3.
For each of $k = 4, 7, 10$, we need to determine whether or not there exists an abc for which there are k abc-BPIs.
For there to exist an abc that has 4, 7 or 10 abc-BPIs, there needs to exist an abc-BPI of the form $abc\,ddd$, because there has to exist a grouping of size 1 in addition to possible groupings of size 3 and 6.
We look systematically at each value of d from $d = 1$ to $d = 9$ and count the number of BPIs attached to a corresponding abc.
In other words, for each of $d = 1$ to $d = 9$, we suppose that $abc\,ddd$ is a BPI for some abc and ask if there are other abc-BPIs of the form $abc\,efg$.
For $d = 1, 5, 7, 8, 9$, there are no other BPIs.
In each case, if $d^3 = e \times f \times g$, we must have $e = f = g = d$.
If $d = 1$ and digits e, f, g satisfy $e \times f \times g = 1$, then $e = f = g = 1$.
If $d = 5$ and digits e, f, g satisfy $e \times f \times g = 5^3 = 125$, then $e = f = g = 5$ since there is no other digit that is a multiple of 5.
If $d = 7$ and digits e, f, g satisfy $e \times f \times g = 7^3 = 343$, then $e = f = g = 7$ since there is no other digit that is a multiple of 7.
If $d = 8$ and digits e, f, g satisfy $e \times f \times g = 8^3 = 512$ and not all of e, f, g are equal to 8, then at least one of the digits must be even and larger than 8, which is not possible.
If $d = 9$ and digits e, f, g satisfy $e \times f \times g = 9^3 = 729$ and not all of e, f, g are equal to 9, then at least one of the digits must be a multiple of 3 and larger than 9, which is not possible.
Consider $d = 2$ and suppose $abc = 222$.
The integer 222 222 is a BPI.
Are there other BPIs of the form $222\,efg$? Yes. If $e \times f \times g = 2^3 = 8$ and e, f, g are digits, then they are all equal to 2, or they are equal to $1, 2, 4$ in some order, or they are equal to $1, 1, 8$ in some order.
Thus, 222 124 is a BPI and rearrangements of the last three digits give a total of 6 BPIs in this grouping.
Also, 222 118 is a BPI and rearrangements of the last three digits give a total of 3 BPIs in this grouping.
Therefore, $abc = 222$ gives $1 + 6 + 3 = 10$ abc-BPIs, so $k = 10$ is a possible total.
In the case $d = 3$, there will be 7 BPIs for any corresponding abc, ending in $333, 139, 193, 319, 391, 913$ so $k = 7$ is a possible total.
Consider $d = 4$ and suppose $abc = 444$.
The integer 444 444 is a BPI.
Are there other BPIs of the form $444\,efg$? Yes. If $e \times f \times g = 4^3 = 64$ and e, f, g are digits, then they are all equal to 4, or they are equal to $1, 8, 8$ in some order, or they are equal to $2, 4, 8$ in some order.
The groupings associated with 444 444, 444 248 and 444 188 have 1, 6 and 3 BPIs, respec-

tively, which gives $1 + 6 + 3 = 10$ BPIs in total.
We know so far that $k = 7$ and $k = 10$ are possible, but have not determined whether or not $k = 4$ is possible.
The only value of d that we have not yet considered is $d = 6$. Since $6 \times 6 \times 6 = 3 \times 8 \times 9$, then there are at least 7 BPIs associated with $d = 6$. In particular, the answer is not 4.
Therefore, $k = 7$ and $k = 10$ are possible numbers and $k = 4$ is not.

We still need to consider $k = 6$ and $k = 9$.
Consider $abc = 579$ and BPIs of the form $579\,def$.
Here, we need $5 \times 7 \times 9 = d \times e \times f$.
Since 5 and 7 are the only digits divisible by 5 and 7, respectively, then two of d, e, f must be 5 and 7, which means that the third digit must be 9.
Therefore, $579\,def$ is a BPI exactly when d, e, f are $5, 7, 9$ in some order.
This means that there are six 579-BPIs, so $k = 6$ is possible.

Consider $abc = 447$ and BPIs of the form $447\,def$.
Here, we need $4 \times 4 \times 7 = d \times e \times f$.
Using a similar argument to the previous case, one of the digits (say f) equals 7, which means that $d \times e = 16$.
Thus, d and e equal 4 and 4 or equal 2 and 8.
Therefore, we have groupings of BPIs that come from $447\,447$ (there are 3) and $447\,287$ (there are 6), which means that there are $3 + 6 = 9$ BPIs in this case.

In conclusion, for each of $k = 6, 7, 9, 10$, there exists an integer abc for which there are exactly k abc-BPIs and for each of $k = 4, 5, 8$, there does not exist an integer abc for which there are exactly k abc-BPIs.