The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2016 Pascal Contest

(Grade 9)

Wednesday, February 24, 2016
(in North America and South America)

Thursday, February 25, 2016
(outside of North America and South America)

Solutions

1. Evaluating, $300 + 2020 + 10001 = 12321$.

   Answer: (E)

2. We evaluate each of the five choices:

$$4^2 = 16 \qquad 4 \times 2 = 8 \qquad 4 - 2 = 2 \qquad \frac{4}{2} = 2 \qquad 4 + 2 = 6$$

   Of these, the largest is $4^2 = 16$.

   Answer: (A)

3. Since the grid is made up of 1×1 squares, then the lengths of the solid line segments, from top to bottom, are 5, 1, 4, 2, 3, and 3.
   The sum of these lengths is $5 + 1 + 4 + 2 + 3 + 3 = 18$.
   Alternatively, we could note that the first and second solid line segments can be combined to form a solid segment of length 6. The same is true with the third and fourth segments, and with the fifth and sixth segments. Thus, the total length is $6 \times 3 = 18$.

   Answer: (D)

4. Since each of the five 1×1 squares has area 1, then the shaded area is 2.
   Since the total area is 5, the percentage that is shaded is $\frac{2}{5} = 0.4 = 40\%$.

   Answer: (D)

5. On a number line, the markings are evenly spaced.
   Since there are 6 spaces between 0 and 30, each space represents a change of $\frac{30}{6} = 5$.
   Since n is 2 spaces to the right of 60, then $n = 60 + 2 \times 5 = 70$.
   Since m is 3 spaces to the left of 30, then $m = 30 - 3 \times 5 = 15$.
   Therefore, $n - m = 70 - 15 = 55$.

   Answer: (C)

6. From the definition, $\begin{array}{c|c} 4 & 5 \\ \hline 2 & 3 \end{array} = 4 \times 3 - 5 \times 2 = 12 - 10 = 2$.

   Answer: (C)

7. Since there are 100 cm in 1 m, then 1 cm is 0.01 m. Thus, 3 cm equals 0.03 m.
   Since there are 1000 mm in 1 m, then 1 mm is 0.001 m. Thus, 5 mm equals 0.005 m.
   Therefore, 2 m plus 3 cm plus 5 mm equals $2 + 0.03 + 0.005 = 2.035$ m.

   Answer: (A)

8. Since $x = 3$ and $y = 2x$, then $y = 2 \times 3 = 6$.
   Since $y = 6$ and $z = 3y$, then $z = 3 \times 6 = 18$.
   Therefore, the average of x, y and z is $\dfrac{x + y + z}{3} = \dfrac{3 + 6 + 18}{3} = 9$.

   Answer: (D)

9. When Team A played Team B, if Team B won, then Team B scored more goals than Team A, and if the game ended in a tie, then Team A and Team B scored the same number of goals.
Therefore, if a team has 0 wins, 1 loss, and 2 ties, then it scored fewer goals than its opponent once (the 1 loss) and the same number of goals as its oppponent twice (the 2 ties).
Combining this information, we see that the team must have scored fewer goals than were scored against them.
In other words, it is not possible for a team to have 0 wins, 1 loss, and 2 ties, and to have scored more goals than were scored against them.

We can also examine choices (A), (B), (D), (E) to see that, in each case, it is possible that the team scored more goals than it allowed.
This will eliminate each of these choices, and allow us to conclude that (C) must be correct.
(A): If the team won 2-0 and 3-0 and tied 1-1, then it scored 6 goals and allowed 1 goal.
(B): If the team won 4-0 and lost 1-2 and 2-3, then it scored 7 goals and allowed 5 goals.
(D): If the team won 4-0, lost 1-2, and tied 1-1, then it scored 6 goals and allowed 3 goals.
(E): If the team won 2-0, and tied 1-1 and 2-2, then it scored 5 goals and allowed 3 goals.
Therefore, it is only the case of 0 wins, 1 loss, and 2 ties where it is not possible for the team to score more goals than it allows.

Answer: (C)

10. *Solution 1*
In the given diagram, we can see 3 of the 6 faces, or $\frac{1}{2}$ of the cube.
The remaining 3 faces (also $\frac{1}{2}$ of the cube) is unshaded.
Of the visible faces, $\frac{1}{2}$ of the area is shaded.
Therefore, the fraction of the total surface area that is shaded is $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Solution 2
Since the cube is $2 \times 2 \times 2$, the area of each face is $2 \times 2 = 4$.
Since a cube has six faces, the total surface area of the cube is $6 \times 4 = 24$.
Each of the three faces that is partially shaded is one-half shaded, since each face is cut into two identical pieces by its diagonal.
Thus, the shaded area on each of these three faces is 2, and so the total shaded area is $3 \times 2 = 6$.
Therefore, the fraction of the total surface area that is shaded is $\frac{6}{24} = \frac{1}{4}$.

Answer: (B)

11. The 7th oblong number is the number of dots in retangular grid of dots with 7 columns and 8 rows.
Thus, the 7th oblong number is $7 \times 8 = 56$.

Answer: (C)

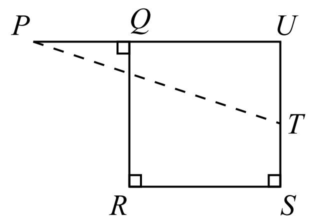
12. Since square $QRST$ has area 36, then its side length is $\sqrt{36} = 6$.
Therefore, $QR = 6$ and $RS = 6$.
Since $PQ = \frac{1}{2}QR$, then $PQ = 3$.
Now rectangle $PRSU$ has height $RS = 6$ and width $PR = PQ + QR = 3 + 6 = 9$.
Therefore, the perimeter of $PRSU$ is $2(6) + 2(9) = 12 + 18 = 30$.

Answer: (B)

13. From the given information, $10x = x + 20$.
Therefore, $9x = 20$ and so $x = \frac{20}{9}$.

Answer: (B)

14. Extend PQ and ST to meet at U.



Since $QUSR$ has three right angles, then it must have four right angles and so is a rectangle.
Thus, $\triangle PUT$ is right-angled at U.
By the Pythagorean Theorem, $PT^2 = PU^2 + UT^2$.
Now $PU = PQ + QU$ and $QU = RS$ so $PU = 4 + 8 = 12$.
Also, $UT = US - ST$ and $US = QR$ so $UT = 8 - 3 = 5$.
Therefore, $PT^2 = 12^2 + 5^2 = 144 + 25 = 169$.
Since $PT > 0$, then $PT = \sqrt{169} = 13$.

Answer: (E)

15. Since $75 = 3 \times 5 \times 5$, we can factor 75 in three different ways:

$$75 = 1 \times 75 = 3 \times 25 = 5 \times 15$$

If $pq = 75$ with p and q integers, then the possible values of p are thus $1, 3, 5, 15, 25, 75$.
The sum of these values is $1 + 3 + 5 + 15 + 25 + 75 = 124$.

Answer: (E)

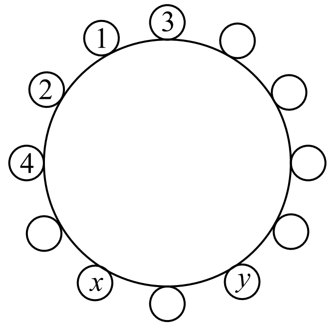
16. From 10 to 99 inclusive, there is a total of 90 integers. (Note that $90 = 99 - 10 + 1$.)
If an integer in this range includes the digit 6, this digit is either the ones (units) digit or the tens digit.
The integers in this range with a ones (units) digit of 6 are 16, 26, 36, 46, 56, 66, 76, 86, 96.
The integers in this range with a tens digit of 6 are 60, 61, 62, 63, 64, 65, 66, 67, 68, 69.
In total, there are 18 such integers. (Notice that 66 is in both lists and $9 + 10 - 1 = 18$.)
Therefore, the probability that a randomly chosen integer from 10 to 99 inclusive includes the digit 6 is $\frac{18}{90} = \frac{1}{5}$.

Answer: (A)

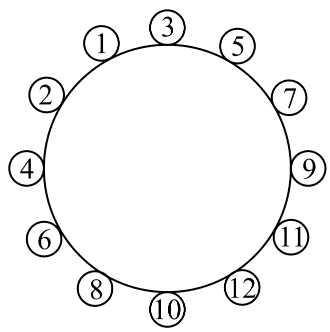
17. Among the list 10, 11, 12, 13, 14, 15, the integers 11 and 13 are prime.
Also, $10 = 2 \times 5$ and $12 = 2 \times 2 \times 3$ and $14 = 2 \times 7$ and $15 = 3 \times 5$.
For an integer N to be divisible by each of these six integers, N must include at least two factors of 2 and one factor each of 3, 5, 7, 11, 13.
Note that $2^2 \times 3 \times 5 \times 7 \times 11 \times 13 = 60\,060$.
(This is the least common multiple of 10, 11, 12, 13, 14, 15.)
To find the smallest six-digit positive integer that is divisible by each of 10, 11, 12, 13, 14, 15, we can find the smallest six-digit positive integer that is a multiple of 60 060.
Note that $1 \times 60\,060 = 60\,060$ and that $2 \times 60\,060 = 120\,120$.
Therefore, the smallest six-digit positive integer that is divisible by each of 10, 11, 12, 13, 14, 15 is 120 120.
The tens digit of this number is 2.

Answer: (C)

18. Because two integers that are placed next to each other must have a difference of at most 2, then the possible neighbours of 1 are 2 and 3.
Since 1 has exactly two neighbours, then 1 must be between 2 and 3.
Next, consider 2. Its possible neighbours are 1, 3 and 4. The number 2 is already a neighbour of 1 and cannot be a neighbour of 3 (since 3 is on the other side of 1). Therefore, 2 is between 1 and 4.
This allows us to update the diagram as follows:



Continuing in this way, the possible neighbours of 3 are 1, 2, 4, 5. The number 1 is already next to 3. Numbers 2 and 4 cannot be next to 3. So 5 must be next to 3.
The possible neighbours of 4 are 2, 3, 5, 6. The number 2 is already next 4. Numbers 3 and 5 cannot be next to 4. So 6 must be next to 4.
Continuing to complete the circle in this way, we obtain:



Note that when the even numbers and odd numbers meet (with 12 and 11) the conditions are still satisfied.
Therefore, $x = 8$ and $y = 12$ and so $x + y = 8 + 12 = 20$.

Answer: (D)

19. Suppose that there were n questions on the test.
Since Chris received a mark of 50% on the test, then he answered $\frac{1}{2}n$ of the questions correctly.
We know that Chris answered 13 of the first 20 questions correctly and then 25% of the remaining questions.
Since the test has n questions, then after the first 20 questions, there are $n - 20$ questions.
Since Chris answered 25% of these $n - 20$ questions correctly, then Chris answered $\frac{1}{4}(n - 20)$ of these questions correctly.
The total number of questions that Chris answered correctly can be expressed as $\frac{1}{2}n$ and also as $13 + \frac{1}{4}(n - 20)$.
Therefore, $\frac{1}{2}n = 13 + \frac{1}{4}(n - 20)$ and so $2n = 52 + (n - 20)$, which gives $n = 32$.
(We can check that if $n = 32$, then Chris answers 13 of the first 20 and 3 of the remaining 12 questions correctly, for a total of 16 correct out of 32.)

Answer: (C)

20. Since $\angle TQP$ and $\angle RQU$ are opposite angles, then $\angle RQU = \angle TQP = x^\circ$.
Similarly, $\angle QRU = \angle VRS = y^\circ$.
Since the angles in a triangle add to 180°, then

$$\angle QUR = 180^\circ - \angle RQU - \angle QRU = 180^\circ - x^\circ - y^\circ$$

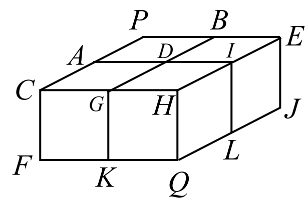
Now $\angle WQP$ and $\angle WQR$ are supplementary, as they lie along a line.
Thus, $\angle WQR = 180^\circ - \angle WQP = 180^\circ - 2x^\circ$.
Similarly, $\angle WRQ = 180^\circ - \angle WRS = 180^\circ - 2y^\circ$.
Since the angles in $\triangle WQR$ add to 180°, then

$$\begin{aligned} 38^\circ + (180^\circ - 2x^\circ) + (180^\circ - 2y^\circ) &= 180^\circ \\ 218^\circ &= 2x^\circ + 2y^\circ \\ x^\circ + y^\circ &= 109^\circ \end{aligned}$$

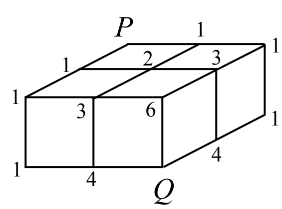
Finally, $\angle QUR = 180^\circ - x^\circ - y^\circ = 180^\circ - (x^\circ + y^\circ) = 180^\circ - 109^\circ = 71^\circ$.

Answer: (A)

21. We label the remaining points on the diagram as shown



There is exactly one path that the squirrel can take to get to each of A, C, F, B, E, and J.
For example, to get to F the squirrel must walk from P to A to C to F.
The number of paths that the squirrel can take to point D is 2, since there is 1 path to each of A and B, and to get to D, the squirrel must go through exactly one of A or B.
Similarly, the number of paths to G is the sum of the number of paths to C and to D (that is, $1 + 2 = 3$), because for the squirrel to get to G, it must walk through exactly one of C or D.
Using this process, we add to the diagram the number of paths to reach each of H, I, K, and L.



Finally, to get to Q, the squirrel must go through exactly one of H, K, or L, so the number of paths to Q is $6 + 4 + 4 = 14$.

Answer: (A)

22. *Solution 1*

Suppose that, when the n students are put in groups of 2, there are g complete groups and 1 incomplete group.
Since the students are being put in groups of 2, an incomplete group must have exactly 1 student in it.
Therefore, $n = 2g + 1$.
Since the number of complete groups of 2 is 5 more than the number of complete groups of 3, then there were $g - 5$ complete groups of 3.
Since there was still an incomplete group, this incomplete group must have had exactly 1 or 2 students in it.
Therefore, $n = 3(g - 5) + 1$ or $n = 3(g - 5) + 2$.
If $n = 2g+1$ and $n = 3(g-5)+1$, then $2g+1 = 3(g-5)+1$ or $2g+1 = 3g-14$ and so $g = 15$.
In this case, $n = 2g + 1 = 31$ and there were 15 complete groups of 2 and 10 complete groups of 3.
If $n = 2g+1$ and $n = 3(g-5)+2$, then $2g+1 = 3(g-5)+2$ or $2g+1 = 3g-13$ and so $g = 14$.
In this case, $n = 2g + 1 = 29$ and there were 14 complete groups of 2 and 9 complete groups of 3.
If $n = 31$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
If $n = 29$, dividing the students into groups of 4 would give 7 complete groups of 4 and 1 incomplete group.
Since the difference between the number of complete groups of 3 and the number of complete groups of 4 is given to be 3, then it must be the case that $n = 31$.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.

Solution 2

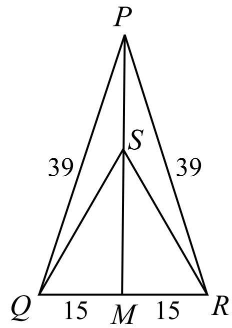
Since the n students cannot be divided exactly into groups of 2, 3 or 4, then n is not a multiple of 2, 3 or 4.
The first few integers larger than 1 that are not divisible by 2, 3 or 4 are 5, 7, 11, 13, 17, 19, 23, 25, 29, 31, and 35.
In each case, we determine the number of complete groups of each size:

n	5	7	11	13	17	19	23	25	29	31	35
# of complete groups of 2	2	3	5	6	8	9	11	12	14	15	17
# of complete groups of 3	1	2	3	4	5	6	7	8	9	10	11
# of complete groups of 4	1	1	2	3	4	4	5	6	7	7	8

Since the number of complete groups of 2 is 5 more than the number of complete groups of 3 which is 3 more than the number of complete groups of 4, then of these possibilities, $n = 31$ works.
In this case, $n^2 - n = 31^2 - 31 = 930$; the sum of the digits of $n^2 - n$ is 12.
(Since the problem is a multiple choice problem and we have found a value of n that satisfies the given conditions and for which an answer is present, then this answer must be correct. Solution 1 shows why $n = 31$ is the only value of n that satisfies the given conditions.)

Answer: (B)

23. Join S to the midpoint M of QR.
Since $\triangle SQR$ is equilateral with side length 30, then $QM = MR = \frac{1}{2}QR = 15$.



Since $\triangle SQR$ is equilateral, then SM is perpendicular to QR.
Since $\triangle PQR$ is isosceles with $PQ = PR$, then PM is also perpendicular to QR.
Since PM is perpendicular to QR and SM is perpendicular to QR, then PM and SM overlap, which means that S lies on PM.
By the Pythagorean Theorem,

$$PM = \sqrt{PQ^2 - QM^2} = \sqrt{39^2 - 15^2} = \sqrt{1521 - 225} = \sqrt{1296} = 36$$

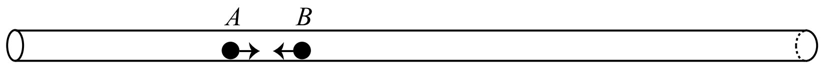
By the Pythagorean Theorem,

$$SM = \sqrt{SQ^2 - QM^2} = \sqrt{30^2 - 15^2} = \sqrt{900 - 225} = \sqrt{675} = 15\sqrt{3}$$

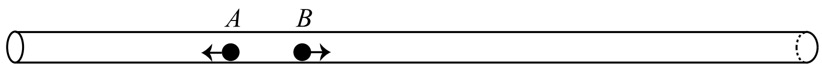
Therefore, $PS = PM - SM = 36 - 15\sqrt{3}$.
Since QM is perpendicular to PS extended, then the area of $\triangle PQS$ is equal to $\frac{1}{2}(PS)(QM)$.
(We can think of PS as the base and QM as the perpendicular height.)
Therefore, the area of $\triangle PQS$ equals $\frac{1}{2}(36 - 15\sqrt{3})(15) \approx 75.14$.
Of the given answers, this is closest to 75.

Answer: (B)

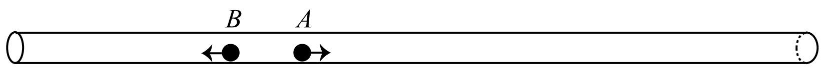
24. Since the rubber balls are very small and the tube is very long (55 m), we treat the balls as points with negligible width.
Since the 10 balls begin equally spaced along the tube with equal spaces before the first ball and after the last ball, then the 10 balls form 11 spaces in the tube, each of which is $\frac{55}{11} = 5$ m long.
When two balls meet and collide, they instantly reverse directions. Before a collision, suppose that ball A is travelling to the right and ball B is travelling to the left.



After this collision, ball A is travelling to the left and ball B is travelling to the right.



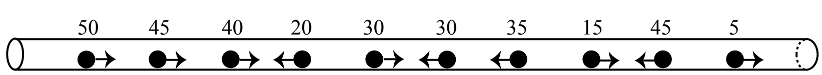
Because the balls have negligible size we can instead pretend that balls A and B have passed through each other and that now ball A is still travelling to the right and ball B is travelling to the left. The negligible size of the balls is important here as it means that we can ignore the fact that the balls will travel slightly further by passing through each other than they would by colliding.



In other words, since one ball is travelling to the left and one is travelling to the right, it actually does not matter how we label them.
This means that we can effectively treat each of the 10 balls as travelling in separate tubes and determine the amount of time each ball would take to fall out of the tube if it travelled in its original direction.
In (A),

- the first ball is 50 m from the right end of the tube, so will take 50 s to fall out
- the second ball is 45 m from the right end of the tube, so will take 45 s to fall out
- the third ball is 40 m from the right end of the tube, so will take 40 s to fall out
- the fourth ball is 20 m from the left end of the tube, so will take 20 s to fall out (note that this ball is travelling to the left)

and so on.
For configuration (A), we can follow the method above and label the amount of time each ball would take to fall out:



We can then make a table that lists, for each of the five configuration, the amount of time, in seconds that each ball, counted from left to right, will take to fall out:

Configuration	#1	#2	#3	#4	#5	#6	#7	#8	#9	#10
(A)	50	45	40	20	30	30	35	15	45	5
(B)	5	45	15	35	30	25	20	40	45	50
(C)	50	10	15	35	30	30	35	15	45	5
(D)	5	45	40	20	30	30	35	40	45	5
(E)	50	10	40	20	30	30	35	15	45	50

Since there are 10 balls, then more than half of the balls will have fallen out when 6 balls have fallen out.
In (A), the balls fall out after 5, 15, 20, 30, 30, 35, 40, 45, 45, and 50 seconds, so 6 balls have fallen out after 35 seconds.
The corresponding times for (B), (C), (D), and (E) are 35, 30, 35, and 35 seconds.
Therefore, the configuration for which it takes the least time for more than half of the balls to fall out is (C).

ANSWER: (C)

25. Since each row in the grid must contain at least one 1, then there must be at least three 1s in the grid.
Since each row in the grid must contain at least one 0, then there must be at least three 0s in the grid. Since there are nine entries in the grid, then there must be at most six 1s in the grid.
Thus, there are three 1s and six 0s, or four 1s and five 0s, or five 1s and four 0s, or six 1s and three 0s.
The number of grids with three 1s and six 0s must be equal to the number of grids with six 1s and three 0s. This is because each grid of one kind can be changed into a grid of the other kind by replacing all of the 0s with 1s and all of the 1s with 0s.
Similarly, the number of grids with four 1s and five 0s will be equal to the number of grids with five 1s and four 0s.
Therefore, we count the number of grids that contain three 1s and the number of grids that contain four 1s, and double our total to get the final answer.

<u>Counting grids that contain three 1s</u>
Since each row must contain at least one 1 and there are only three 1s to use, then there must be exactly one 1 in each row.
Since each column must also contain a 1, then the three rows must be $\boxed{1\ 0\ 0}$, $\boxed{0\ 1\ 0}$, and $\boxed{0\ 0\ 1}$ in some order.
There are thus 3 choices for the first row.
For each of these choices, there are 2 choices for the second row. The first and second rows completely determine the third row.
Therefore, there are $3 \times 2 = 6$ (or $3 \times 2 \times 1 = 6$) configurations for the grid.
We note that each of these also includes at least one 0 in each row and in each column, as desired.

<u>Counting grids that contain four 1s</u>
Since each row must contain at least one 1 and there are four 1s to use, then there must be two 1s in one row and one 1 in each of the other two rows. This guarantees that there is at least one 0 in each row.
Suppose that the row containing two 1s is $\boxed{1\ 1\ 0}$.
One of the remaining rows must have a 1 in the third column, so must be $\boxed{0\ 0\ 1}$.
The remaining row could be any of $\boxed{1\ 0\ 0}$, $\boxed{0\ 1\ 0}$, and $\boxed{0\ 0\ 1}$.
We note that in any combination of these rows, each column will contain at least one 0 as well.
With the rows $\boxed{1\ 1\ 0}$, $\boxed{0\ 0\ 1}$, and $\boxed{0\ 0\ 1}$, there are 3 arrangements.
This is because there are 3 choices of where to put the row $\boxed{1\ 1\ 0}$, and then the remaining two rows are the same and so no further choice is possible.
With the rows $\boxed{1\ 1\ 0}$, $\boxed{0\ 0\ 1}$, and $\boxed{0\ 1\ 0}$, there are 6 arrangements, using a similar argument to the counting in the "three 1s" case above.
Similarly, with rows $\boxed{1\ 1\ 0}$, $\boxed{0\ 0\ 1}$, and $\boxed{1\ 0\ 0}$, there are 6 arrangements.
So there are $3 + 6 + 6 = 15$ configurations that include the row $\boxed{1\ 1\ 0}$.

Using similar arguments, we can find that there are 15 configurations that include the row $\boxed{\texttt{1 0 1}}$ and 15 configurations that include the row $\boxed{\texttt{0 1 1}}$.
Therefore, there are $3 \cdot 15 = 45$ configurations that contain four 1s.

Finally, by the initial comment, this means that there are $2(6 + 45) = 102$ configurations.

ANSWER: (D)