The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2016 Gauss Contests

(Grades 7 and 8)

Wednesday, May 11, 2016
(in North America and South America)

Thursday, May 12, 2016
(outside of North America and South America)

Solutions

Centre for Education in Mathematics and Computing Faculty and Staff

Gauss Contest Committee

Grade 7

1. Evaluating, $333 + 33 + 3 = 366 + 3 = 369$.

Answer: (D)

2. The day on which Tanner received the most text messages will be the day with the tallest corresponding bar.
Thus, Tanner received the most text messages on Friday.

Answer: (A)

3. *Solution 1*
A number is a multiple of 7 if it is the result of multiplying 7 by an integer.
Of the answers given, only 77 results from multiplying 7 by an integer, since $77 = 7 \times 11$.

Solution 2
A number is a multiple of 7 if the result after dividing it by 7 is an integer.
Of the answers given, only 77 results in an integer after dividing by 7, since $77 \div 7 = 11$.

Answer: (C)

4. A positive fraction is larger than $\frac{1}{2}$ if its denominator is less than two times its numerator.
Of the answers given, $\frac{4}{7}$ is the only fraction in which the denominator, 7, is less than 2 times its numerator, 4 (since $2 \times 4 = 8$).
Therefore, $\frac{4}{7}$ is larger than $\frac{1}{2}$.

Answer: (C)

5. Rolling the cube does not change the size of the painted triangle.
For this reason, we can eliminate answer (A).
Rolling the cube does not change the number of painted triangles.
For this reason, we can eliminate answers (D) and (E).
Rolling the cube does not change the orientation of the painted triangle with respect to the face of the cube that it is painted on.
For this reason, we can eliminate answer (C).
Of the given answers, the cube shown in (B) is the only cube which could be the same as the cube that was rolled.

Answer: (B)

6. The measure of the three angles in any triangle add to 180°.
Since two of the angles measure 25° and 70°, then the third angle in the triangle measures $180^\circ - 25^\circ - 70^\circ = 85^\circ$.
The measure of the third angle in the triangle is 85°.

Answer: (A)

7. Each of the 30 pieces of fruit in the box is equally likely to be chosen. Since there are 10 oranges in the box, then the probability that the chosen fruit is an orange is $\frac{10}{30}$ or $\frac{1}{3}$.

Answer: (D)

8. *Solution 1*
Since Alex pays \$2.25 to take the bus, then 20 trips on the bus would cost Alex $20 \times \$2.25 = \$45$.
Since Sam pays \$3.00 to take the bus, then 20 trips on the bus would cost Sam $20 \times \$3.00 = \$60$.
If they each take the bus 20 times, then in total Alex would pay $\$60 - \$45 = \$15$ less than Sam.

Solution 2
Since Alex pays \$2.25 to take the bus, and Sam pays \$3.00 to take the bus, then Alex pays $\$3.00 - \$2.25 = \$0.75$ less than Sam each time they take the bus.
If they each take the bus 20 times, then in total Alex would pay $20 \times \$0.75 = \$15$ less than Sam.

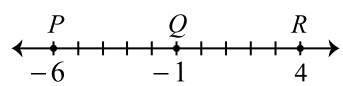
ANSWER: (C)

9. *Solution 1*
Travelling at a constant speed of 85 km/h, the entire 510 km trip would take Carrie $510 \div 85 = 6$ hours.
Since Carrie is halfway through the 510 km trip, then the remainder of the trip will take her half of the total trip time or $6 \div 2 = 3$ hours.

Solution 2
Carrie is halfway through a 510 km trip, and so she has half of the distance or $510 \div 2 = 255$ km left to travel.
Since Carrie travels at a constant speed of 85 km/h, then it will take her $255 \div 85 = 3$ hours longer to complete the trip.

ANSWER: (E)

10. Since Q is halfway between P and R, then the distance between P and Q is equal to the distance between Q and R.
The distance between P and Q is $-1 - (-6) = -1 + 6 = 5$.
Since P is 5 units to the left of Q, then R is 5 units to the right of Q.
That is, R is located at $-1 + 5 = 4$ on the number line.



ANSWER: (A)

11. In the diagram, there are 4 rows of octagons and each row contains 5 octagons.
Therefore, the total number of octagons in the diagram is $4 \times 5 = 20$.
In the diagram, there are 3 rows of squares and each row contains 4 squares.
Therefore, the total number of squares in the diagram is $3 \times 4 = 12$.
The ratio of the number of octagons to the number of squares is $20 : 12$ or $5 : 3$.

ANSWER: (E)

12. The sum of the units column is $Q + Q + Q = 3Q$.
Since Q is a single digit, and $3Q$ ends in a 6, then the only possibility is $Q = 2$.
Then $3Q = 3 \times 2 = 6$, and thus there is no carry over to the tens column.
The sum of the tens column becomes $2 + P + 2 = P + 4$, since $Q = 2$.
Since P is a single digit, and $P + 4$ ends in a 7, then the only possibility is $P = 3$.
Then $P + 4 = 3 + 4 = 7$, and thus there is no carry over to the hundreds column.
We may verify that the sum of the hundreds column is $3 + 3 + 2 = 8$, since $P = 3$ and $Q = 2$.
The value of $P + Q$ is $3 + 2 = 5$, and the final sum is shown.

$$\begin{array}{r} 322 \\ 332 \\ +\,222 \\ \hline 876 \end{array}$$

ANSWER: (B)

13. Since a cube is a rectangular prism, its volume is equal to the area of its base, $l \times w$, multiplied by its height, h.
A cube has edges of equal length and so $l = w = h$.
Thus, the volume of a cube is the product of three equal numbers.
The volume of the larger cube is $64\,\text{cm}^3$ and $64 = 4 \times 4 \times 4$, so the length of each edge of the larger cube is $4\,\text{cm}$.
The smaller cube has edges that are half the length of the edges of the larger cube, or $2\,\text{cm}$.
The volume of the smaller cube is $2 \times 2 \times 2 = 8\,\text{cm}^3$.

Answer: (C)

14. Ahmed could choose from the following pairs of snacks: apple and orange, apple and banana, apple and granola bar, orange and banana, orange and granola bar, or banana and granola bar.
Therefore, there are 6 different pairs of snacks that Ahmed may choose.

Answer: (D)

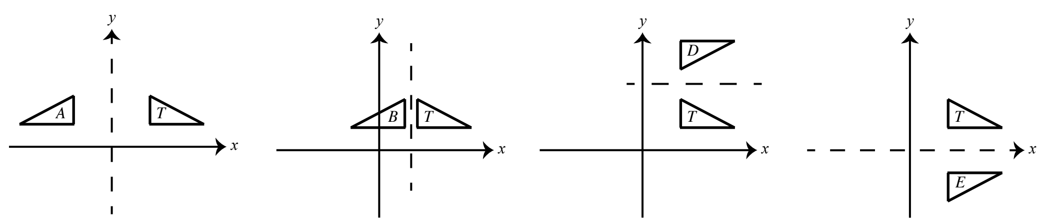
15. Sophia did push-ups for 7 days (an odd number of days), and on each day she did an equal number of push-ups more than the day before (5 more).
Therefore, the number of push-ups that Sophia did on the middle day (day 4) is equal to the average number of push-ups that she completed each day.
Sophia did 175 push-ups in total over the 7 days, and thus on average she did $175 \div 7 = 25$ push-ups each day.
Therefore, on day 4 Sophia did 25 push-ups, and so on day 5 she did $25 + 5 = 30$ push-ups, on day 6 she did $30 + 5 = 35$ push-ups and on the last day she did $35 + 5 = 40$ push-ups.
(Note: We can check that $10 + 15 + 20 + 25 + 30 + 35 + 40 = 175$, as required.)

Answer: (E)

16. Since $\square = \triangle + \triangle + \triangle$, then by adding a $\blacklozenge$ to each side we get $\square + \blacklozenge = \blacklozenge + \triangle + \triangle + \triangle$.
Since $\square + \blacklozenge = \blacklozenge + \triangle + \triangle + \triangle$, then by adding a $\triangle$ to each side we get that $\square + \blacklozenge + \triangle = \blacklozenge + \triangle + \triangle + \triangle + \triangle$.
(Can you explain why each of the other answers is not equal to $\square + \blacklozenge + \triangle$?)

Answer: (B)

17. Each of the following four diagrams shows the image of triangle T after its reflection in the dotted line.



Thus, each of the triangles labelled A, B, D, and E is a single reflection of triangle T in some line.
The triangle labelled C is the only triangle that cannot be a reflection of triangle T.

Answer: (C)

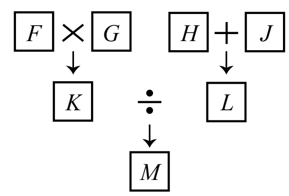
18. The mean (average) of the set of six numbers is 10, and so the sum of the set of six numbers is $6 \times 10 = 60$.
If the number 25 is removed from the set, the sum of the set of the remaining five numbers is $60 - 25 = 35$.
The mean (average) of the remaining set of five numbers is $35 \div 5 = 7$.

Answer: (B)

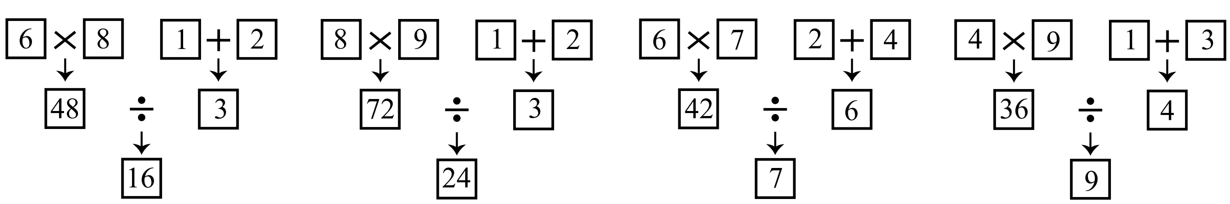
19. The shaded and unshaded sections of the ribbon have equal length.
Since there are 5 such sections, then each shaded and unshaded section has length equal to $\frac{1}{5}$ or $\frac{3}{15}$ of the length of the ribbon.
All measurements which follow are made beginning from the left end of the ribbon.
Point A is located 3 sections from the left end of the ribbon, or at a point $3 \times \frac{3}{15} = \frac{9}{15}$ along the length of the ribbon.
Point D is located 4 sections from the left end of the ribbon, or at a point $4 \times \frac{3}{15} = \frac{12}{15}$ along the length of the ribbon.
All points are equally spaced, and so points B and C divide the unshaded section between points A and D into 3 equal lengths.
Since A is located $\frac{9}{15}$ of the ribbon length from the left end, and D is located $\frac{12}{15}$ of the ribbon length from the left end, then B is located at $\frac{10}{15}$ of the ribbon length, and C is located at $\frac{11}{15}$ of the ribbon length.
Thus, if Suzy makes a vertical cut at point C, the portion of the ribbon to the left of C will be $\frac{11}{15}$ of the size of the original ribbon.
We note that no point is located more than 2 sections from the right end of the ribbon.
That is, no point is located more than $2 \times \frac{3}{15} = \frac{6}{15}$ along the length of the ribbon when measured from the right end, and so measurements are taken from the left end of the ribbon.

Answer: (C)

20. We begin by naming the boxes as shown to the right.



Of the five answers given, the integer which cannot appear in box M is 20. Why?
Since boxes F and G contain different integers, the maximum value that can appear in box K is $8 \times 9 = 72$.
Since boxes H and J contain different integers, the minimum value that can appear in box L is $1 + 2 = 3$.
Next, we consider the possibilities if 20 is to appear in box M.
If 3 appears in box L (the minimum possible value for this box), then box K must contain 60, since $60 \div 3 = 20$.
However, there are no two integers from 1 to 9 whose product is 60 and so there are no possible integers which could be placed in boxes F and G so that the product in box K is 60.
If any integer greater than or equal to 4 appears in box L, then box K must contain at least $4 \times 20 = 80$.
However, the maximum value that can appear in box K is 72.
Therefore, there are no possible integers from 1 to 9 which can be placed in boxes F, G, H, and J so that 20 appears in box M.
The diagrams below demonstrate how each of the other four answers can appear in box M.

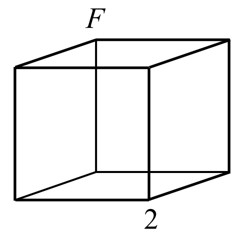


Answer: (D)

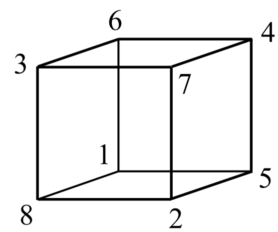
21. Line segment PQ is vertical if Q is chosen from the points in the column in which P lies.
This column contains 9 points other than P which could be chosen to be Q so that PQ is vertical.
Line segment PQ is horizontal if Q is chosen from the points in the row in which P lies.
This row contains 9 points other than P which could be chosen to be Q so that PQ is horizontal.
Each of these 9 points is different from the 9 points in the column containing P.
Thus, there are $9 + 9 = 18$ points which may be chosen to be Q so that PQ is vertical or horizontal.
Since there are a total of 99 points to choose Q from, the probability that Q is chosen so that PQ is vertical or horizontal is $\frac{18}{99}$ or $\frac{2}{11}$.

Answer: (A)

22. First we choose to label one of the vertices 2, and then label the vertex that is farthest away from this vertex F, as shown.
(Can you explain why the vertex labelled F is the vertex that is farthest away from the vertex labelled 2?)
Each of the other six vertices of the cube lie on one of the three faces on which the vertex labelled 2 lies.



We note that the vertex labelled F is the only vertex which does not lie on a face on which the vertex labelled 2 lies.
From the six given lists, we consider those lists in which the number 2 appears.
These are: $(1, 2, 5, 8)$, $(2, 4, 5, 7)$, and $(2, 3, 7, 8)$.
Thus, the vertices labelled $1, 5$ and 8 lie on a face with the vertex labelled 2, as do the vertices labelled $4, 7$ and 3.
The only vertex label not included in this list is 6.
Thus, the vertex labelled 6 is the only vertex which does not lie on a face on which the vertex labelled 2 lies.
Therefore, the correct labelling for vertex F, the farthest vertex from the vertex labelled 2, is 6.
One possible labelling of the cube is shown.



Answer: (D)

23. *Solution 1*

Let the letter R represent a red marble, and the letter B represent a blue marble.
On her first draw, Angie may draw RR, RB or BB.

Case 1: Angie draws RR or RB on her first draw
If Angie draws RR or RB on her first draw, then she discards the R and the three remaining marbles in the jar are RBB.
On her second draw, Angie may draw RB or BB.
If she draws RB, then she discards the R and the two remaining marbles in the jar are BB.
Since there are no red marbles remaining, it is not possible for the final marble to be red in this case.
If on her second draw Angie instead draws BB, then she discards a B and the two remaining marbles in the jar are RB.
When these are both drawn on her third draw, the R is discarded and the final marble is blue.
Again in this case it is not possible for the final marble to be red.
Thus, if Angie draws RR or RB on her first draw, the probability that the final marble is red is zero.

Case 2: Angie draws BB on her first draw
If Angie draws BB on her first draw, then she discards a B and the three remaining marbles in the jar are RRB.
On her second draw, Angie may draw RR or RB.
If she draws RR or RB, then she discards one R and the two remaining marbles in the jar are RB.
When these are both drawn on her third draw, the R is discarded and the final marble is blue.
In this case it is not possible for the final marble to be red.
Thus, if Angie draws BB on her first draw, the probability that the final marble is red is zero.

Therefore, under the given conditions of drawing and discarding marbles, the probability that Angie's last remaining marble is red is zero.

Solution 2

Let the letter R represent a red marble, and the letter B represent a blue marble.
If the final remaining marble is R, then the last two marbles must include at least one R.
That is, the last two marbles must be RB or RR.
If the last two marbles are RB, then when they are drawn from the jar, the R is discarded and the B would remain.
Thus it is not possible for the final marble to be R if the final two marbles are RB.
So the final remaining marble is R only if the final two marbles are RR.
If the final two marbles are RR, then the last three marbles are BRR (since there are only two Rs in the jar at the beginning).
However, if the final three marbles are BRR, then when Angie draws two of these marbles from the jar, at least one of the marbles must be R and therefore one R will be discarded leaving BR as the final two marbles in the jar.
That is, it is not possible for the final two marbles in the jar to be RR.
The only possibility that the final remaining marble is R occurs when the final two marbles are RR, but this is not possible.
Therefore, under the given conditions of drawing and discarding marbles, the probability that Angie's last remaining marble is red is zero.

ANSWER: (E)

24. We begin by showing that each of 101, 148, 200, and 621 can be expressed as the sum of two or more consecutive positive integers.

$$\begin{aligned} 101 &= 50+51 \\ 148 &= 15+16+17+18+19+20+21+22 \\ 200 &= 38+39+40+41+42 \\ 621 &= 310+311 \end{aligned}$$

We show that 512 cannot be expressed a sum of two or more consecutive positive integers. This will tell us that one of the five numbers in the list cannot be written in the desired way, and so the answer is (B).

Now, 512 cannot be written as the sum of an odd number of consecutive positive integers.
Why is this? Suppose that 512 equals the sum of p consecutive positive integers, where $p > 1$ is odd.
Since p is odd, then there is a middle integer m in this list of p integers.
Since the numbers in the list are equally spaced, then m is the average of the numbers in the list.
(For example, the average of the 5 integers $6, 7, 8, 9, 10$ is 8.)
But the sum of the integers equals the average of the integers (m) times the number of integers (p). That is, $512 = mp$.
Now $512 = 2^9$ and so does not have any odd divisors larger than 1.
Therefore, 512 cannot be written as mp since m and p are positive integers and $p > 1$ is odd.
Thus, 512 is not the sum of an odd number of consecutive positive integers.

Further, 512 cannot be written as the sum of an even number of consecutive positive integers.
Why is this? Suppose that 512 equals the sum of p consecutive positive integers, where $p > 1$ is even.
Since p is even, then there is not a single middle integer m in this list of p integers, but rather two middle integers m and $m + 1$.
Since the numbers in the list are equally spaced, then the average of the numbers in the list is the average of m and $m + 1$, or $m + \frac{1}{2}$.
(For example, the average of the 6 integers $6, 7, 8, 9, 10, 11$ is $8\frac{1}{2}$.)
But the sum of the integers equals the average of the integers $(m + \frac{1}{2})$ times the number of integers (p). That is, $512 = (m + \frac{1}{2})p$ and so $2(512) = 2(m + \frac{1}{2})p$ or $1024 = (2m + 1)p$.
Now $1024 = 2^{10}$ and so does not have any odd divisors larger than 1.
Therefore, 1024 cannot be written as $(2m+1)p$ since m and p are positive integers and $2m+1 > 1$ is odd.
Thus, 512 is not the sum of an even number of consecutive positive integers.

Therefore, 512 is not the sum of any number of consecutive positive integers.
A similar argument shows that every power of 2 cannot be written as the sum of any number of consecutive positive integers.
Returning to the original question, exactly one of the five numbers in the original list cannot be written in the desired way, and so the answer is (B).

Answer: (B)

25. Consider the diagonal lines that begin on the left edge of the triangle and move downward to the right.
The first number in the n^{th} diagonal line is n, and it lies in the n^{th} horizontal row.
For example, the first number in the 3^{rd} diagonal line $(\mathbf{3}, 6, 9, 12, \ldots)$ is 3 and it lies in the 3^{rd} horizontal row $(\mathbf{3}, 4, 3)$.
The second number in the n^{th} diagonal line is $n + n$ or $2n$ and it lies in the horizontal row numbered $n + 1$.
The third number in the n^{th} diagonal line is $n + n + n$ or $3n$ and it lies in the horizontal row numbered $n + 2$ (each number lies one row below the previous number in the diagonal line).
Following this pattern, the m^{th} number in the n^{th} diagonal line is equal to $m \times n$ and it lies in the horizontal row numbered $n + (m - 1)$.
The table below demonstrates this for $n = 3$, the 3^{rd} diagonal line.

m	m^{th} Diagonal Number	Horizontal Row Number
1	3	3
2	$2(3) = 6$	$3 + 1 = 4$
3	$3(3) = 9$	$3 + 2 = 5$
4	$4(3) = 12$	$3 + 3 = 6$
5	$5(3) = 15$	$3 + 4 = 7$
$\vdots$	$\vdots$	$\vdots$
m	$m \times n$	$3 + (m - 1)$

The number 2016 lies in some diagonal line(s).
To determine which diagonal lines 2016 lies in, we express 2016 as a product $m \times n$ for positive integers m and n.
Further, if $2016 = m \times n$, then 2016 appears in the triangle in position m in diagonal line n, and lies in the horizontal row numbered $n + m - 1$.
We want the horizontal row in which 2016 first appears, and so we must find positive integers m and n so that $m \times n = 2016$ and $n + m$ (and therefore $n + m - 1$) is as small as possible.
In the table below, we summarize the factor pairs (m, n) of 2016 and the horizontal row number $n + m - 1$ in which each occurence of 2016 appears.

Factor Pair (m, n)	Horizontal Row Number $n + m - 1$
$(1, 2016)$	2016
$(2, 1008)$	1009
$(3, 672)$	674
$(4, 504)$	507
$(6, 336)$	341
$(7, 288)$	294
$(8, 252)$	259
$(9, 224)$	232
$(12, 168)$	179

Factor Pair (m, n)	Horizontal Row Number $n + m - 1$
$(14, 144)$	157
$(16, 126)$	141
$(18, 112)$	129
$(21, 96)$	116
$(24, 84)$	107
$(28, 72)$	99
$(32, 63)$	94
$(36, 56)$	91
$(42, 48)$	89

(Note: By recognizing that when $m \times n = 2016$, the sum $n + m$ is minimized when the positive difference between m and n is minimized, we may shorten the work shown above.)
We have included all possible pairs (m, n) so that $m \times n = 2016$ in the table above.
We see that 2016 will appear in 18 different locations in the triangle.
However, the first appearance of 2016 occurs in the horizontal row numbered 89.

Answer: (E)

Grade 8

1. Evaluating, $444 - 44 - 4 = 400 - 4 = 396$.

   Answer: (A)

2. *Solution 1*
   The fraction $\frac{4}{5}$ is equal to $4 \div 5$ or 0.8.
   Solution 2
   Since $\frac{4}{5}$ is equal to $\frac{8}{10}$, then $\frac{4}{5} = 0.8$.

   Answer: (B)

3. Reading from the graph, we summarize the number of hours that Stan worked each day in the table below.

Day	Monday	Tuesday	Wednesday	Thursday	Friday
Number of Hours	2	0	3	1	2

   Therefore, Stan worked a total of $2 + 0 + 3 + 1 + 2 = 8$ hours on the project.

   Answer: (C)

4. Written numerically, three tenths plus four thousandths is $\frac{3}{10} + \frac{4}{1000}$ which is equal to $\frac{300}{1000} + \frac{4}{1000} = \frac{304}{1000} = 0.304$.

   Answer: (C)

5. Folds occur along the five edges between adjoining faces in the figure shown.
   Consider the face numbered 3 as being the bottom face of the completed cube.
   First, fold upward along the four edges of the face numbered 3 (the edges between 3 and 5, 3 and 4, 3 and 6, and 3 and 2).
   After folding upward, the faces numbered $2, 5, 4$, and 6 become the four vertical faces of the cube.
   The final fold occurs along the edge between the faces numbered 1 and 2.
   The face numbered 1 becomes the top face of the cube after this fold.
   Since the bottom face is opposite the top face, then the face numbered 3 is opposite the face numbered 1.

   Answer: (B)

6. Side PR is horizontal and so the y-coordinate of P is equal to the y-coordinate of R, or -2.
   Side PQ is vertical and so the x-coordinate of P is equal to the x-coordinate of Q, or -11.
   Therefore, the coordinates of P are $(-11, -2)$.

   Answer: (D)

7. A rectangle with a width of 2 cm and a length of 18 cm has area $2 \times 18 = 36\,\text{cm}^2$.
   The area of a square with side length s cm is $s \times s\,\text{cm}^2$.
   The area of the square is also $36\,\text{cm}^2$ and since $6 \times 6 = 36$, then the side length of the square, s, is 6 cm.

   Answer: (A)

8. From the list $3, 4, 5, 6, 7, 8, 9$, only $3, 5$ and 7 are prime numbers.
   The numbers $4, 6, 8$, and 9 are composite numbers.
   The ratio of the number of prime numbers to the number of composite numbers is $3 : 4$.

   Answer: (A)

9. Since 10% of 200 is $\frac{10}{100} \times 200 = 10 \times 2 = 20$, and 20% of 100 is $\frac{20}{100} \times 100 = 20$, then 10% of 200 is equal to 20% of 100.

Answer: (C)

10. The circumference of a circle with radius r is equal to $2\pi r$.
When $2\pi r = 100\pi$, $r = \frac{100\pi}{2\pi} = 50\,\text{cm}$.

Answer: (C)

11. In equilateral triangle QRS, each angle is equal in measure and so $\angle SQR = 60^\circ$.
Since $\angle PQR = 90^\circ$, then $\angle PQS = \angle PQR - \angle SQR = 90^\circ - 60^\circ = 30^\circ$.
In isosceles triangle PQS, $\angle QPS = \angle PQS = 30^\circ$.
Therefore, $\angle QPR = \angle QPS = 30^\circ$.

Answer: (E)

12. We try each of the five options:

(A): $3 + 5 \times 7 + 9 = 3 + 35 + 9 = 47$
(B): $3 + 5 + 7 \times 9 = 3 + 5 + 63 = 71$
(C): $3 \times 5 \times 7 - 9 = 15 \times 7 - 9 = 105 - 9 = 96$
(D): $3 \times 5 \times 7 + 9 = 15 \times 7 + 9 = 105 + 9 = 114$
(E): $3 \times 5 + 7 \times 9 = 15 + 63 = 78$

Therefore, the correct operations are, in order, $\times, +, \times$.

Answer: (E)

13. Ahmed could choose from the following pairs of snacks: apple and orange, apple and banana, apple and granola bar, orange and banana, orange and granola bar, or banana and granola bar.
Therefore, there are 6 different pairs of snacks that Ahmed may choose.

Answer: (D)

14. One soccer ball and one soccer shirt together cost \$100.
So then two soccer balls and two soccer shirts together cost $2 \times \$100 = \$200$.
Since we are given that two soccer balls and three soccer shirts together cost \$262, then \$200 added to the cost of one soccer shirt is \$262.
Thus, the cost of one soccer shirt is $\$262 - \$200 = \$62$, and the cost of one soccer ball is $\$100 - \$62 = \$38$.

Answer: (A)

15. The map's scale of $1 : 600\,000$ means that a 1 cm distance on the map represents an actual distance of 600 000 cm.
So then 2 cm measured on the map represents an actual distance of $2 \times 600\,000 = 1\,200\,000$ cm or 12 000 m or 12 km.
The actual distance between Gausstown and Piville is 12 km.

Answer: (A)

16. The mean (average) of the set of six numbers is 10, and so the sum of the set of six numbers is $6 \times 10 = 60$.
If the number 25 is removed from the set, the sum of the set of the remaining five numbers is $60 - 25 = 35$.
The mean (average) of the remaining set of five numbers is $35 \div 5 = 7$.

Answer: (B)

17. The positive integers between 10 and 2016 which have all of their digits the same are: $11, 22, 33, 44, 55, 66, 77, 88, 99, 111, 222, 333, 444, 555, 666, 777, 888, 999$, and 1111.
To be divisible by 3, the sum of the digits of the positive integer must equal a multiple of 3.
From the list above, the only 2-digit numbers whose digit sum is a multiple of 3 are $33, 66$ and 99 (with digit sums of 6, 12 and 18, respectively).
(We may verify that each of the other digit sums, $2, 4, 8, 10, 14$, and 16 are not multiples of 3.)
A 3-digit positive integer with all digits equal to d has digit sum $d + d + d = 3d$ (which is a multiple of 3).
Thus, all 3-digit positive integers with equal digits are divisible by 3.
That is, all 9 of the 3-digit integers listed above are divisible by 3.
Finally, the number 1111 has digit sum 4 and thus is not divisible by 3.
There are $3 + 9 = 12$ positive integers between 10 and 2016, having all of their digits the same, that are divisible by 3.

Answer: (B)

18. Joe used $\frac{3}{8}$ of the gas in the tank after travelling 165 km, and so he used $\frac{3}{8} \div 3 = \frac{1}{8}$ of the gas in the tank after travelling $165 \div 3 = 55$ km.
Since Joe used $\frac{1}{8}$ of the gas in the tank to travel 55 km, he used all the gas in the tank to travel $55 \times 8 = 440$ km.
If Joe has already travelled 165 km, then he can travel another $440 - 165 = 275$ km before the gas tank is completely empty.

Answer: (E)

19. The first scale shows that 2 $\bigcirc$'s balance 6 $\square$'s and so 1 $\bigcirc$ balances 3 $\square$'s.
Thus, we may eliminate answer (C).
The second scale shows that 2 $\bigcirc$'s and 6 $\square$'s balance 4 $\triangle$'s and so 1 $\bigcirc$ and 3 $\square$'s balance 2 $\triangle$'s.
Thus, we may eliminate answer (B).
Since 1 $\bigcirc$ and 3 $\square$'s balance 2 $\triangle$'s and 1 $\bigcirc$ balances 3 $\square$'s, then 6 $\square$'s balance 2 $\triangle$'s or 3 $\square$'s balance 1 $\triangle$.
Thus, we may eliminate answer (E).
Since 1 $\bigcirc$ balances 3 $\square$'s, and 1 $\triangle$ balances 3 $\square$'s, then 1 $\bigcirc$ balances 1 $\triangle$.
Thus, we may eliminate answer (A).
Finally, we are left with answer (D).
Since 1 $\bigcirc$ balances 3 $\square$'s and 1 $\triangle$ balances 3 $\square$'s, then 1 $\bigcirc$ and 1 $\triangle$ balance 6 $\square$'s, not 4 $\square$'s.
Thus, answer (D) is the only answer which is not true.

Answer: (D)

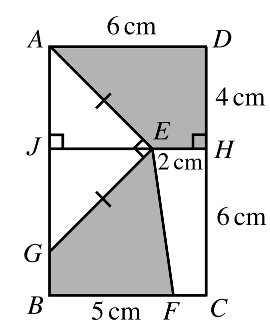
20. Points D and C have equal y-coordinates, -3, and so side DC is parallel to the x-axis and has length $3 - (-2) = 5$.
Points B and C have equal x-coordinates, 3, and so side BC is parallel to the y-axis and has length $9 - (-3) = 12$.
That is, in $\triangle BCD$, sides DC and BC are perpendicular or $\angle BCD = 90^\circ$ with $BC = 9$ and $DC = 5$.
Using the Pythagorean Theorem, $BD^2 = DC^2 + BC^2 = 5^2 + 12^2$ and so $BD^2 = 25 + 144 = 169$ or $BD = \sqrt{169} = 13$ (since $BD > 0$).

Answer: (A)

21. If the ten thousands digits of the two numbers differ by more than 1, then the two numbers will differ by more than 10 000. (For example, a number of the form $5x\,xxx$ is at least 50 000 and a number of the form $3x\,xxx$ is less than 40 000 so these numbers differ by more than 10 000.)
Since all of the given answers are less than 1000 and since the two ten thousands digits cannot be equal, then the ten thousands digits must differ by 1. We will determine the exact ten thousands digits later, so we let the smaller of the two ten thousands digits be d and the larger be D.
To make the difference between $Dx\,xxx$ and $dx\,xxx$ as small as possible, we try to simultaneously make $Dx\,xxx$ as close to $D0\,000$ as possible and $dx\,xxx$ as close to $d9\,999$ as possible while using all different digits.
In other words, we try to make $Dx\,xxx$ as small as possible and $dx\,xxx$ as large as possible while using all different digits.
To make $Dx\,xxx$ as small as possible, we use the smallest possible digits in the places with the highest value.
Since all of the digits must be different, then the minimum possible value of $Dx\,xxx$ is $D0\,123$.
To make $dx\,xxx$ as large as possible, we use the largest possible digits in the places with the highest value.
Since all of the digits must be different, then the maximum possible value of $dx\,xxx$ is $d9\,876$.
Since we have made $Dx\,xxx$ as small as possible and $dx\,xxx$ as large as possible and used completely different sets of digits, then doing these two things will make their difference as small as possible, assuming that there are digits remaining to assign to D and d that differ by 1.
The digits that have not been used are 5 and 4; thus, we set $D = 5$ and $d = 4$.
This gives numbers 50 123 and 49 876.
Their difference is $50\,123 - 49\,876 = 247$, which is the minimum possible difference.

Answer: (C)

22. We find the area of the shaded region by determining the area of the unshaded region and subtracting this from the total area of the rectangle.



We begin by extending HE to J on side AB, as shown.
Since HE is perpendicular to DH, then HJ is perpendicular to DH.
Since DH is parallel to AJ, then HJ is also perpendicular to AJ, and so $ADHJ$ is a rectangle.
Since $ADHJ$ is a rectangle, then $AJ = DH = 4$ cm.
Also, $AD = JH = 6$ cm.
Since $JH = 6$ cm, then $HE + EJ = 6$ cm or $2\text{ cm} + EJ = 6$ cm and so $EJ = 4$ cm.
Since $\triangle AEG$ is isosceles with $AE = GE$, then altitude EJ divides base AG into two equal lengths.
Since $AJ = 4$ cm, then $GJ = AJ = 4$ cm.
Therefore, $\triangle AEG$ has base $AG = 8$ cm and height $EJ = 4$ cm and so its area is $\frac{1}{2}(8)(4) = 16\text{ cm}^2$.
Since $BC = AD = 6$ cm, then $BF + CF = 6$ cm or $5\text{ cm} + CF = 6$ cm and so $CF = 1$ cm.
In quadrilateral $EHCF$, sides HE and CF are both perpendicular to DC, and so they are parallel to each other.
That is, quadrilateral $EHCF$ is a trapezoid with parallel sides $EH = 2$ cm, $CF = 1$ cm, and height $HC = 6$ cm (since HC is perpendicular to both HE and CF).
The area of trapezoid $EHCF$ is $\frac{1}{2}(6)(2 + 1) = 9\text{ cm}^2$.
The total area of the shaded region is found by subtracting the area of $\triangle AEG$ and the area of trapezoid $EHCF$ from the area of rectangle $ABCD$.

The area of rectangle $ABCD$ is $6(6+4) = 6(10) = 60\,\text{cm}^2$, and so the total area of the shaded region is $60 - 16 - 9 = 35\,\text{cm}^2$.

ANSWER: (D)

23. For Zeus to arrive at the point $(1056, 1007)$, he must travel up and to the right from his starting point $(0, 0)$.
More specifically, Zeus will need to make at least 1056 moves to the right (R).
Since Zeus cannot move in the same direction twice in a row, then no two moves R can be next to each other. In other words, there must be at least one move between each pair of Rs.
Since there are 1056 moves R and there must be at least one move between each pair, then there are at least 1055 more moves (at least one between the 1^{st} and 2^{nd} R, between the 2^{nd} and 3^{rd} R, and so on).
Therefore, at least $1056 + 1055 = 2111$ moves are needed.
We will show that there is actually a sequence of 2111 moves that obey the given rules and that take Zeus to $(1056, 1007)$.
For Zeus to end at a point with y-coordinate 1007, he will need to make at least 1007 moves up (U).
So we put one U between the 1^{st} and 2^{nd} R, between the 2^{nd} and 3^{rd} R, and so on up to between the 1007^{th} and the 1008^{th} R.
This leaves us with $1055 - 1007 = 48$ spaces between Rs to fill.
We want to fill these spaces with moves that do not affect Zeus' position (since we already have him moving 1056 moves R and 1007 moves U) and so that none of the moves are R (since the spaces to be filled are between adjacent moves R).
We can do this by making the first 24 of these moves U and the remaining 24 of these moves down (D). This results in Zeus moving an additional 24 units up but then an additional 24 moves down, which is no net change in position.
We have shown that Zeus needs at least 2111 moves to get to $(1056, 1007)$ and that he can actually get to $(1056, 1007)$ in 2111 moves following the given rules, so the smallest number of moves that he needs is 2111.

ANSWER: (D)

24. When two integers are multiplied together, the final two digits (the tens digit and the units digits) of the product are determined by the final two digits of each of the two numbers that are multiplied.
This is true since the place value of any digit contributes to its equal place value (and possibly also to a greater place value) in the product.
That is, the hundreds digit of each number being multiplied contributes to the hundreds digit (and possibly to digits of higher place value) in the product.
Thus, to determine the tens digit of any product, we need only consider the tens digits and the units digits of each of the two numbers that are being multiplied.
For example, to determine the final two digits of the product 1215×603 we consider the product $15 \times 03 = 45$. We may verify that the tens digit of the product $1215 \times 603 = 732\,645$ is indeed 4 and the units digit is indeed 5.
Since $3^5 = 243$, then the final two digits of $3^{10} = 3^5 \times 3^5 = 243 \times 243$ are given by the product $43 \times 43 = 1849$ and thus are 49.
Since the final two digits of 3^{10} are 49 and $3^{20} = 3^{10} \times 3^{10}$, then the final two digits of 3^{20} are given by $49 \times 49 = 2401$, and thus are 01.
Then $3^{40} = 3^{20} \times 3^{20}$ ends in 01 also (since $01 \times 01 = 01$).
Further, 3^{20} multiplied by itself 100 times, or $(3^{20})^{100} = 3^{2000}$ also ends with 01.

Since 3^{10} ends with 49 and 3^5 ends with 43, then $3^{15} = 3^{10} \times 3^5$ ends with $49 \times 43 = 2107$ and thus has final two digits 07.
This tells us that $3^{16} = 3^{15} \times 3^1$ ends with $07 \times 03 = 21$.
Finally, $3^{2016} = 3^{2000} \times 3^{16}$ and thus ends in $01 \times 21 = 21$, and so the tens digit of 3^{2016} is 2.

ANSWER: (B)

25. We begin by adding variables to some of the blanks in the grid to make it easier to refer to specific entries:

				18
	43	f	g	h
		40		j
		k		m
x	n	p	26	q

Since the numbers in each row form an arithmetic sequence and the numbers in each column form an arithmetic sequence, we will refer in several sequences to the *common difference*.
Let the common difference between adjacent numbers as we move down column 3 be d.
Therefore, $k = 40 + d$ and $p = k + d = 40 + 2d$.
Also, $40 = f + d$, or $f = 40 - d$.
Moving from left to right along row 2, the common difference between adjacent numbers must be $(40 - d) - 43 = -3 - d$.
Therefore,

$$g = f + (-3 - d) = (40 - d) + (-3 - d) = 37 - 2d$$
$$h = g + (-3 - d) = (37 - 2d) + (-3 - d) = 34 - 3d$$

This gives:

				18
	43	$40 - d$	$37 - 2d$	$34 - 3d$
		40		j
		$40 + d$		m
x	n	$40 + 2d$	26	q

Moving from the top to the bottom of column 5, the common difference between adjacent numbers can be found by subtracting 18 from $34 - 3d$.
That is, the common difference between adjacent numbers in column 5 is $(34-3d)-18 = 16-3d$.
Moving down column 5, we can see that

$$j = (34 - 3d) + (16 - 3d) = 50 - 6d$$
$$m = (50 - 6d) + (16 - 3d) = 66 - 9d$$
$$q = (66 - 9d) + (16 - 3d) = 82 - 12d$$

In each case, we have added the common difference to the previous number to obtain the next number.

This gives the following updated grid:

				18
	43	$40-d$	$37-2d$	$34-3d$
		40		$50-6d$
		$40+d$		$66-9d$
x	n	$40+2d$	26	$82-12d$

In row 5, the difference between the fourth and third entries must equal the difference between the fifth and fourth entries.
In other words,

$$\begin{aligned} 26-(40+2d) &= (82-12d)-26 \\ -14-2d &= 56-12d \\ 10d &= 70 \\ d &= 7 \end{aligned}$$

We can then substitute $d=7$ into our grid to obtain:

				18
	43	33	23	13
		40		8
		47		3
x	n	54	26	-2

This allows us to determine the value of x by moving along row 5.
We note that $26-54=-28$ so the common difference in row 5 is -28.
Therefore, $n+(-28)=54$ or $n=54+28=82$.
Similarly, $x+(-28)=n=82$ and so $x=82+28=110$.
Therefore, the sum of the digits of the value of $x=110$ is $1+1+0=2$.

Answer: (B)