The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2016 Fryer Contest

Wednesday, April 13, 2016
(in North America and South America)

Thursday, April 14, 2016
(outside of North America and South America)

Solutions

1. (a) The total score for School A is found by adding the scores of the four students competing from the school.
Therefore, the total score for School A is $12 + 8 + 10 + 6 = 36$.

(b) Since the total score for School A is 36, then the total score for School B is also 36.
The scores of the four students competing from School B are $17, 5, 7$, and x, and so $17 + 5 + 7 + x = 36$ or $29 + x = 36$ and so $x = 7$.

(c) The score for Student 4 at School C, z, is twice that of Student 3 at School C, y.
Thus, $z = 2y$.
The scores of the four students competing from School C are $9, 15, y$, and z.
Since the total score for School C is also 36, and $z = 2y$, then $9 + 15 + y + 2y = 36$ or $24 + 3y = 36$ or $3y = 12$, and so $y = 4$.
Therefore, the score for Student 3 at School C is 4, and the score for Student 4 at School C is $2(4) = 8$.

2. (a) Every 2 seconds, Esther takes 5 steps and each of her steps is 0.4 m long.
Therefore, in 2 seconds Esther travels a distance of $5 \times 0.4 = 2$ m.

(b) *Solution 1*
Every 2 seconds, Paul takes 5 steps and each of his steps is 1.2 m long.
Therefore, every 2 seconds Paul travels a distance of $5 \times 1.2 = 6$ m.
Since Paul travels 6 m every 2 seconds, his speed is $6 \div 2 = 3$ m/s.

Solution 2
Paul takes 5 steps every 2 seconds and so Paul takes 2.5 steps every second.
The length of each of Paul's steps is 1.2 m, and so every second Paul travels a distance of $2.5 \times 1.2 = 3$ m.
Thus, Paul travels at a speed of 3 m/s.

(c) *Solution 1*
Paul travels at a speed of 3 m/s, and so in 120 seconds (2 minutes), Paul travels a distance of $3 \times 120 = 360$ m.
Esther travels 2 m every 2 seconds and so she travels at a speed of 1 m/s.
In 120 seconds, Esther travels a distance of $1 \times 120 = 120$ m.
If Paul and Esther start a race at the same time, then after 2 minutes, Paul will be $360 - 120 = 240$ m ahead of Esther.

Solution 2
Each of Paul's steps is $1.2 - 0.4 = 0.8$ m longer than each of Esther's steps.
Every 2 seconds, Paul and Esther each take 5 steps, and so in $2 \times 60 = 120$ seconds (2 minutes), Paul and Esther each take $5 \times 60 = 300$ steps.
If Paul and Esther start a race at the same time, then after 2 minutes Paul will be $300 \times 0.8 = 240$ m ahead of Esther.

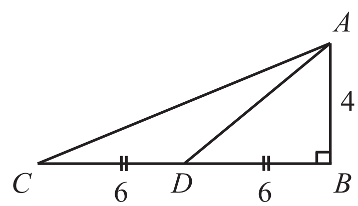
Solution 3
Paul travels at a speed of 3 m/s, and Esther travels 2 m every 2 seconds, so she travels at a speed of 1 m/s.
This means that in 1 second, Paul travels 3 m and Esther travels 1 m.
Every second, Paul travels $3 - 1 = 2$ m farther than Esther.
If Paul and Esther start a race at the same time, then after 120 seconds (2 minutes), Paul will be $120 \times 2 = 240$ m ahead of Esther.

(d) *Solution 1*
Esther travels 2 m every 2 seconds or 1 m/s, and so in 180 seconds (3 minutes), Esther travels $180 \times 1 = 180$ m.
Paul travels at a speed of 3 m/s, which is 2 m/s faster than the speed at which Esther travels.
Therefore, every second, Paul travels 2 m farther than Esther travels.
Since Esther begins the race 180 m ahead of Paul, it will take Paul $180 \div 2 = 90$ s to catch Esther.

Solution 2
Esther travels 2 m every 2 seconds or 1 m/s, and so in 180 seconds (3 minutes), Esther travels $180 \times 1 = 180$ m.
Each of Paul's steps is $1.2 - 0.4 = 0.8$ m longer than each of Esther's steps.
Since Paul and Esther each step at the same rate (5 steps every 2 seconds), then it will take Paul $180 \div 0.8 = 225$ steps to catch Esther.
Paul takes 5 steps every 2 seconds, and so it will take Paul $\frac{225}{5} \times 2 = 45 \times 2 = 90$ s to catch Esther.

3. (a) *Solution 1*
In $\triangle ABC$, AD is a median and so D is the midpoint of BC.
Since $BC = 12$ and D is the midpoint of BC, then $CD = \frac{12}{2} = 6$.
In $\triangle ACD$, base CD has length 6, and corresponding height AB has length 4. (Since $\angle ABC = 90^\circ$, AB is the height of $\triangle ACD$ even though AB is outside $\triangle ACD$.)
Thus, $\triangle ACD$ has area $\frac{1}{2}(6)(4) = 12$.

Solution 2
In $\triangle ABC$, AD is a median and so D is the midpoint of BC.
Since $BC = 12$ and D is the midpoint of BC, then $CD = DB = 6$.



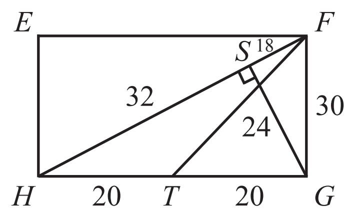
In $\triangle ABD$, $AB = 4$, $DB = 6$, and $\angle ABD = 90^\circ$, and so $\triangle ABD$ has area $\frac{1}{2}(6)(4) = 12$.
Similarly, $\triangle ABC$ has area $\frac{1}{2}(12)(4) = 24$, and so the area of $\triangle ACD$ is the area of $\triangle ABC$ minus the area of $\triangle ABD$, or $24 - 12 = 12$.

Solution 3
In $\triangle ABC$, $AB = 4$, $BC = 12$, and $\angle ABC = 90^\circ$, and so $\triangle ABC$ has area $\frac{1}{2}(12)(4) = 24$.
A median of $\triangle ABC$ divides the triangle into two equal areas. Why?

> In $\triangle ABC$, AD is a median and so D is the midpoint of BC.
> Therefore, $\triangle ACD$ and $\triangle ABD$ have equal bases ($CD = BD$).
> Further, the height of $\triangle ABD$ is equal to the height of $\triangle ACD$ (both are AB).
> Thus, $\triangle ACD$ and $\triangle ABD$ have equal bases and equal heights.
> Since the area of each triangle equals one-half times the base times the height, then $\triangle ABD$ and $\triangle ACD$ have equal areas and so median AD divides $\triangle ABC$ into equal areas.

Since $\triangle ABC$ has area 24, then $\triangle ACD$ has area $\frac{24}{2} = 12$.

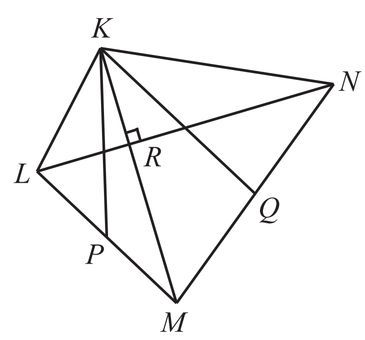
(b) *Solution 1*
In $\triangle FSG$, $FS = 18$, $SG = 24$, and $\angle FSG = 90^\circ$.
Thus, by the Pythagorean Theorem, $FG = \sqrt{18^2 + 24^2} = \sqrt{324 + 576} = \sqrt{900} = 30$ (since $FG > 0$).
Since S is on FH so that $FS = 18$ and $SH = 32$, then $FH = FS + SH = 18 + 32 = 50$.
In $\triangle FGH$, $FH = 50$, $FG = 30$, and $\angle FGH = 90^\circ$.
Thus, by the Pythagorean Theorem, $GH = \sqrt{50^2 - 30^2} = \sqrt{2500 - 900} = \sqrt{1600} = 40$ (since $GH > 0$).
In $\triangle FGH$, FT is a median and so T is the midpoint of GH.
In $\triangle FHT$, base $HT = \frac{40}{2} = 20$, and height $FG = 30$. (Since $\angle FGH = 90^\circ$, FG is the height of $\triangle FHT$ even though FG is outside $\triangle FHT$.)



Thus, $\triangle FHT$ has area $\frac{1}{2}(20)(30) = 300$.

Solution 2
Since S is on FH so that $FS = 18$ and $SH = 32$, then $FH = FS + SH = 18 + 32 = 50$.
In $\triangle FGH$, base $FH = 50$, and height $SG = 24$ (since SG is perpendicular to FH, SG is a height of $\triangle FGH$).
Thus, $\triangle FGH$ has area $\frac{1}{2}(50)(24) = 600$.
The median of a triangle divides the area of the triangle in half.
(Solution 3 to (a) shows an example of why a median divides a triangle's area in half.)
Since FT is a median of $\triangle FGH$, then the area of $\triangle FHT = \frac{600}{2} = 300$.

(c) We use the notation $|\triangle KLM|$ to represent the area of $\triangle KLM$, $|KPMQ|$ to represent the area of $KPMQ$, and so on.
In $\triangle KLM$, KP is a median and so $2|\triangle KPM| = |\triangle KLM|$.
(Solution 3 to (a) shows an example of why a median divides a triangle's area in half.)
In $\triangle KMN$, KQ is a median and so $2|\triangle KMQ| = |\triangle KMN|$.



Therefore,

$$|KLMN| = |\triangle KLM| + |\triangle KMN| = 2|\triangle KPM| + 2|\triangle KMQ|$$

and

$$|KPMQ| = |\triangle KPM| + |\triangle KMQ|$$

which tells us that $|KLMN| = 2|KPMQ|$.
Since $|KPMQ| = 63$, then $|KLMN| = 2|KPMQ| = 2(63) = 126$.
Now $|KLMN| = |\triangle KRL| + |\triangle LRM| + |\triangle KRN| + |\triangle NRM|$.
Each of these four triangles is right-angled.
Since $KR = x$ and $LR = 6$, then $|\triangle KRL| = \frac{1}{2}x(6) = 3x$.
Since $LR = 6$ and $RM = 2x + 2$, then $|\triangle LRM| = \frac{1}{2}(6)(2x + 2) = 6x + 6$.
Since $KR = x$ and $RN = 12$, then $|\triangle KRN| = \frac{1}{2}x(12) = 6x$.
Since $RN = 12$ and $RM = 2x + 2$, then $|\triangle NRM| = \frac{1}{2}(12)(2x + 2) = 12x + 12$.
Therefore, $126 = 3x+(6x+6)+6x+(12x+12)$ or $126 = 27x+18$ or $27x = 108$ and so $x = 4$.

4. (a) Because the entries from one column do not affect the possible entries in another column, then the smallest possible sum of the numbers in a row equals the sum of the smallest possible number in each column.

Column	Possible entries	Smallest possible entry
B	$1, 2, 3, \ldots, 13, 14, 15$	1
I	$16, 17, 18, \ldots, 28, 29, 30$	16
N	$0, 31, 32, 33, \ldots, 43, 44, 45$	0
G	$46, 47, 48, \ldots, 58, 59, 60$	46
O	$61, 62, 63, \ldots, 73, 74, 75$	61

Thus, the smallest possible sum of the numbers in a row on a BINGO card is equal to $1 + 16 + 0 + 46 + 61$ or 124. (This sum can only occur in the middle row, since the entry 0 can only occur in the middle of the square.)

(b) *Solution 1*

From part (a), the smallest possible sum of the numbers in a row is 124.
This minimum row sum occurs in the 3^{rd} row so that the number in column N is 0.
(Note that if we do not use the 3^{rd} row, then the smallest number that can occur in column N is 31, and thus the minimum possible row sum in any row other than the 3^{rd} is $1 + 16 + 31 + 46 + 61 = 124 + 31 = 155$.)
The minimum sum of the numbers in a diagonal is also 124 since the smallest possible number in each column (including the middle entry 0) may be used in a diagonal sum.
The minimum row sum and the minimum diagonal sum each use the numbers $1, 16, 0, 46$, and 61.
However, the number 1 cannot occur in both the 3^{rd} row and in the diagonal since every BINGO card is filled with twenty-five different integers.
The smallest two numbers that can appear in the 3^{rd} row and in the diagonal in column B are 1 and 2.
Similarly, the smallest two numbers that can appear in the 3^{rd} row and in the diagonal in column I are 16 and 17.
In column N, the 3^{rd} row and the diagonal intersect and share the smallest number 0.
In column G, the number 46 cannot appear in both the 3^{rd} row and in the diagonal, and so the smallest two numbers that can appear in the 3^{rd} row and in the diagonal in column G are 46 and 47.
Similarly, the smallest two numbers that can appear in the 3^{rd} row and in the diagonal in column O are 61 and 62.
Therefore, in any BINGO card, the combined list of numbers in a diagonal sum and a row sum must include 10 numbers that are at least as large as 1, 2, 16, 17, 0, 0, 46, 47, 61, 62.
The sum of these numbers is $1 + 2 + 16 + 17 + 0 + 0 + 46 + 47 + 61 + 62 = 252$.

Thus, if Carrie's BINGO card has a row and a diagonal each with the same sum, then this sum must be at least one-half of this total; that is, the minimum such sum is $\frac{1}{2}(252) = 126$. One BINGO card showing this minimum equal row and diagonal sum of 126 is possible, is shown below.

B	I	N	G	O
1				
	17			
2	16	0	47	61
			46	
				62

We confirm that the row and diagonal sums are $2+16+0+47+61 = 1+17+0+46+62 = 126$, as claimed.
Each of the empty spaces in the card may be filled with any of the appropriate numbers not yet used.

Solution 2

From part (a), the smallest possible sum of the numbers in a row is 124.
This minimum row sum occurs in the 3^{rd} row so that the number in column N is 0.
(Note that if we do not use the 3^{rd} row, then the smallest number that can occur in column N is 31, and thus the minimum possible row sum in any row other than the 3^{rd} is $1 + 16 + 31 + 46 + 61 = 124 + 31 = 155$.)
The minimum sum of the numbers in a diagonal is also 124 since the smallest possible number in each column (including the middle entry 0) could be used in a diagonal sum.
The minimum row sum and the minimum diagonal sum each must use the numbers $1, 16, 0, 46, 61$.
However, the number 1 cannot occur in both the 3^{rd} row and in a diagonal, since 1 cannot occur twice in the B column.
Similarly, 16 cannot occur in both the 3^{rd} row and in the diagonal, since 16 cannot occur twice in the I column. As well, 46 and 61 cannot occur in both the 3^{rd} row and in a diagonal. Therefore, it is not possible for a BINGO card to have both the 3^{rd} row and a diagonal sum to 124.
A row sum of 126 and a diagonal sum of 126 is possible, as the following BINGO card shows:

B	I	N	G	O
1				
	16			
2	17	0	46	61
			47	
				62

We confirm that the row and diagonal sums are $2 + 17 + 0 + 46 + 61 = 126$ and $1 + 16 + 0 + 47 + 62 = 126$, as claimed. Each of the empty spaces in the card may be filled with any of the appropriate numbers not yet used to create a full BINGO card.

Is it possible to have a BINGO card with a row sum of 125 and a diagonal sum of 125?
If not, then the smallest possible number that can be both a row sum and a diagonal sum will be 126.
Consider the smallest possible diagonal sum $1 + 16 + 0 + 46 + 61 = 124$.

Since 1, 16, 0, 46, 61 are the smallest possible entries in each column, then a diagonal sum of 125 can only be created by replacing exactly one of the four integers 1, 16, 46, 61 by the integer that is one larger.
In particular, the possible diagonals with a sum of 125 are

$$2+16+0+46+61=125 \qquad 1+17+0+46+61=125$$

$$1+16+0+47+61=125 \qquad 1+16+0+46+62=125$$

For the same reason, these are also the possible 3^{rd} rows with a sum of 125.
It is not possible for a BINGO card to have one of these sums as a diagonal sum and a different one of these sums as a 3^{rd} row sum, since each pair of these sums has more numbers than just the 0 in common. (For example, 2+16+0+46+61 and 1+16+0+47+61 share 16, 0 and 61, and the number 16 cannot appear in the I column in both a diagonal and the 3^{rd} row.)

Therefore, the smallest possible number that can be both a diagonal sum and a row sum is 126.

(c) *Solution 1*
The maximum possible sum of the numbers in the 3^{rd} row and in the diagonal is $15+30+0+60+75=180$.
We need the sum of the numbers in the 3^{rd} row and the sum of the numbers in the diagonal to both be 177.
We determine the number of ways in which this can be done by starting with the largest possible numbers and reducing these numbers to reduce the sums to 177.
Thus, we call the missing numbers in the 3^{rd} row $15-W, 30-X, 60-Y, 75-Z$ for some integers W, X, Y, Z and the missing numbers in the diagonal $15-w, 30-x, 60-y, 75-z$ for some integers w, x, y, z, as shown:

B	I	N	G	O
$15-w$	23	35	47	65
5	$30-x$	31	52	63
$15-W$	$30-X$	0	$60-Y$	$75-Z$
11	20	40	$60-y$	69
9	18	38	48	$75-z$

Since the numbers in the first column are between 1 and 15, inclusive, then $w \geq 0$ and $W \geq 0$. Similarly, each of x, X, y, Y, z, Z are greater than or equal to 0.
Since the numbers in the 3^{rd} row have a sum of 177, then

$$(15-W)+(30-X)+0+(60-Y)+(75-Z)=177$$

or $180-(W+X+Y+Z)=177$ and so $W+X+Y+Z=3$.
Similarly, we have $w+x+y+z=3$.
Since the B column cannot contain repeated numbers, then $15-w$ and $15-W$ cannot be equal, which means that we cannot have $w=W$. Also, $x \neq X$ and $y \neq Y$ and $z \neq Z$.

The number of BINGO cards with the desired property is equal to the number of ways that we can choose non-negative integers w, x, y, z, W, X, Y, Z with the correct sums and so that no two numbers in the same column are equal.

Since W, X, Y, Z are integers that are at least 0, then they must be $3, 0, 0, 0$ in some order or $2, 1, 0, 0$ in some order or $1, 1, 1, 0$ in some order:

> For the sum of non-negative integers to be 3, no single integer can be larger than 3. If one integer is 3, the rest are 0. If one integer is 2, then we must have one 1 and the rest equal to 0. If no integers equal 3 or 2, then we must have three 1s.

Similarly, w, x, y, z must be $3, 0, 0, 0$ in some order or $2, 1, 0, 0$ in some order or $1, 1, 1, 0$ in some order.

Note that since none of the values can be larger than 3, then the missing entries in the B, I, G, and O columns are at least 12, 27, 57, and 72, respectively, so cannot duplicate existing entries.

To count the BINGO cards, we now count the possible combinations of values for W, X, Y, Z and w, x, y, z.

In this discussion, we call w and W *corresponding positions*. Similarly, x and X, y and Y, and z and Z will be called corresponding positions.

<u>Case 1: W, X, Y, Z are $3, 0, 0, 0$ in some order</u>

(i) w, x, y, z cannot be $3, 0, 0, 0$. If it were, then at least two pairs of corresponding positions will equal 0, which would mean that at least two columns of the card would contain the same number twice. (For example, if $W = 3$, $X = 0$, $Y = 0$, $Z = 0$ and $w = 0$, $x = 0$, $y = 3$, $z = 0$, then $X = x = 0$ and $Z = z = 0$ which means that the I column will include 30 twice and the O column will include 75 twice.)

(ii) w, x, y, z cannot be $2, 1, 0, 0$ because at least one pair of corresponding positions will equal 0, which means that at least one column of the BINGO card will contain the same number twice.

(iii) w, x, y, z could be $1, 1, 1, 0$ in some order. In how many ways can this happen?
Since W, X, Y, Z are $3, 0, 0, 0$ in some order, then there are 4 possible positions in which the 3 can go. The remaining three positions must be 0.
Looking at w, x, y, z, the 0 must go in the position corresponding to the 3 (since there cannot be two 0s in corresponding positions) and so the 1s go in the remaining positions.
In total, this means that there are 4 possible ways in which this can happen.

<u>Case 2: W, X, Y, Z are $2, 1, 0, 0$ in some order</u>

(i) w, x, y, z cannot be $3, 0, 0, 0$ as we saw in Case 1(ii).

(ii) w, x, y, z could be $2, 1, 0, 0$ in some order.
Looking at W, X, Y, Z, there are 4 possible positions for the 2. For each of these positions, there are 3 possible positions for the 1. The 0s go in the remaining two positions.
Looking at w, x, y, z, the 0s must go in the positions that correspond to the 2 and 1 among W, X, Y, Z. This means that there are 2 possible positions for the 2 and then the 1 is placed in the last position.
Overall, there are $4 \cdot 3 \cdot 2 = 24$ ways in which this can be done.

(iii) w, x, y, z could be $1, 1, 1, 0$ in some order.
In how many ways can this happen?
Looking at W, X, Y, Z, there are 4 possible positions for the 2. For each of these positions, there are 3 possible positions for the 1. The 0s go in the remaining two positions.
Looking at w, x, y, z, the 0 must go in the corresponding position to the 1 among W, X, Y, Z, since there cannot be two 1s in this position. The positions of the 1s are then completely determined.
Overall, there are $4 \cdot 3 = 12$ ways in which this can be done.

Case 3: W, X, Y, Z are $1, 1, 1, 0$ in some order

(i) w, x, y, z could be $3, 0, 0, 0$ in some order. As we saw in Case 1(iii), there are 4 ways in which this can happen.

(ii) w, x, y, z could be $2, 1, 0, 0$ in some order. As we saw in Case 2(iii), there are 12 ways in which this can happen.

(iii) w, x, y, z cannot be $1, 1, 1, 0$ in some order because at least two pairs of corresponding variables will equal 1.

In total, there are thus $4 + 24 + 12 + 4 + 12 = 56$ ways in which W, X, Y, Z, w, x, y, z can be determined.
Each set of values of these variables gives a BINGO card with the desired property, and so there are 56 ways of completing the BINGO card so that the sum of the numbers in the diagonal and in the 3^{rd} row are each 177.

Solution 2
The maximum possible sum of the numbers in the 3^{rd} row and in the diagonal is $15 + 30 + 0 + 60 + 75 = 180$.
We require both the sum of the numbers in the 3^{rd} row and the sum of the numbers in the diagonal to be 177.
Since 177 is 3 less than the maximum possible sum of 180, then any of the missing numbers in the given BINGO card can be at most 3 less than the largest number that can appear in columns B, I, G, and O (column N is fixed at 0).
That is, the smallest number that can appear in the 3^{rd} row and in the diagonal of column B is $15 - 3 = 12$.
Similarly, the smallest numbers that can appear in the 3^{rd} row and in the diagonal of columns I, G and O are 27, 57 and 72, respectively.
Thus, the missing numbers in the given BINGO card must be chosen from:

$12, 13, 14, 15$ in column B,
$27, 28, 29, 30$ in column I,
$57, 58, 59, 60$ in column G, and
$72, 73, 74, 75$ in column O.

(Note that these numbers do not already appear in the given BINGO card and so they each may be chosen to fill blank spaces.)
There are three different methods in which the maximum sum, 180, can be decreased by exactly 3 to give a row or diagonal sum of 177.
From the lists above, we may choose:

- the smallest number from one of the four columns (this number is 3 less than the largest), and choose the largest number from each of the remaining three columns. For example we could choose the smallest number from column B, 12, and the largest numbers from the remaining columns, $30, 60, 75$, since $12 + 30 + 60 + 75 = 177$, or
- the largest number from one of the four columns, and choose the second largest number from each of the remaining three columns. For example we could choose the largest number from column B, 15, and the second largest numbers from the remaining columns, $29, 59, 74$, since $15 + 29 + 59 + 74 = 177$, or
- the largest numbers from two of the four columns, and choose the second largest number from one of the remaining two columns and the third largest number from the final column. For example we could choose the largest numbers from columns B

and I, 15 and 30, and the second largest number from column G, 59, and the third largest number from column O, 73, since $15 + 30 + 59 + 73 = 177$.

These are the only three methods in which we can decrease the maximum row and diagonal sum of 180 by exactly 3 to give a row or diagonal sum of 177.
We restate these three methods by considering the following table:

	B	I	G	O
P	15	30	60	75
Q	14	29	59	74
R	13	28	28	73
S	12	27	57	72

The three methods for achieving a row or diagonal sum of 177 are:

(1) choose 1 number from group S (the numbers $12, 27, 57, 72$), and 3 numbers from group P, or

(2) choose 1 number from group P, and 3 numbers from group Q, or

(3) choose 2 numbers from group P, 1 number from group Q, and 1 number from group R.

We require both the 3^{rd} row sum and the diagonal sum to be 177.
Thus, we must use two of the above methods simultaneously and we must ensure that no number appears twice in any given column.
Which pairs of combinations may be chosen from the three methods listed?
If we fill the blanks in the 3^{rd} row of the BINGO card using method 1, then we cannot fill the blanks of the diagonal using method 1 since each application of method 1 requires that we use 3 different numbers from group P, and there are only 4 numbers to choose from in any of the groups.
Further, if we fill the blanks in the 3^{rd} row of the BINGO card using method 1, then we cannot fill the blanks of the diagonal using method 3 since this would require 5 numbers from group P.
Therefore, if the 3^{rd} row is filled using method 1, then the diagonal must be filled using method 2.
Similarly, we cannot fill the blanks in the 3^{rd} row of the BINGO card using method 2 and at the same time use method 2 to fill the blanks in the diagonal.
We also cannot fill the blanks in the 3^{rd} row of the BINGO card using method 3 and at the same time use method 1 to fill the blanks in the diagonal.
All other combinations of methods are possible and they are summarized in the table:

Method used to fill the 3^{rd} row	Method used to fill the diagonal
1: $SPPP$	2: $PQQQ$
2: $PQQQ$	1: $SPPP$
2: $PQQQ$	3: $PPQR$
3: $PPQR$	2: $PQQQ$
3: $PPQR$	3: $PPQR$

Finally, to count the number of ways to complete the BINGO card, we must count the number of ways to choose numbers that satisfy each of the five combinations listed above.

First Combination: $SPPP$ in the $3rd$ row and $PQQQ$ in the diagonal

$SPPP$ can occur in 4 ways in the 3^{rd} row:
Select one of the 4 columns (B, I, G, O) in which to choose the group S number, and the remaining 3 columns are each filled with their group P number.

To complete the diagonal in this same BINGO card using $PQQQ$, we recognize that the group P number must occur in the same column in which the group S number occured in the 3^{rd} row, because each of the other 3 group P numbers are already in the 3^{rd} row.
That is, there is only 1 choice for the placement of the group P number, and each of the remaining 3 columns in the diagonal will be filled with their group Q number.
So there are 4 ways to select the numbers for the 3^{rd} row and then only 1 way to select the numbers for the diagonal, and thus there are $4 \times 1 = 4$ ways to complete the BINGO card using this first combination.

<u>Second Combination: $PQQQ$ in the $3rd$ row and $SPPP$ in the diagonal</u>
The counting here is identical to that of the first combination above, with the roles of the 3^{rd} row and diagonal reversed.
Thus, there are 4 ways in which the BINGO card can be completed using this second combination.

<u>Third Combination: $PQQQ$ in the $3rd$ row and $PPQR$ in the diagonal</u>
$PQQQ$ can occur in 4 ways in the 3^{rd} row:

> Select one of the 4 columns (B, I, G, O) in which to choose the group P number, and the remaining 3 columns are each filled with their group Q number.

To complete the diagonal in this same BINGO card using $PPQR$, we recognize that the group Q number must occur in the same column in which the group P number occured in the 3^{rd} row, because each of the other 3 group Q numbers are already in the 3^{rd} row.
Next we must fill the remaining 3 columns with their respective PPR numbers.
These can be ordered in 3 different ways (PPR, PRP, and RPP) and so there are 3 ways to fill the remaining 3 columns in the diagonal.
So there are 4 ways to select the numbers for the 3^{rd} row and then 1×3 ways to select the numbers for the diagonal, and thus there are $4 \times 3 = 12$ ways to complete the BINGO card using this third combination.

<u>Fourth Combination: $PPQR$ in the $3rd$ row and $PQQQ$ in the diagonal</u>
The counting here is identical to that of the third combination above, with the roles of the 3^{rd} row and diagonal reversed.
Thus there are 12 ways in which the BINGO card can be completed using this fourth combination.

<u>Fifth Combination: $PPQR$ in the $3rd$ row and $PPQR$ in the diagonal</u>
$PPQR$ can occur in 12 ways in the 3^{rd} row:

> Select one of the four columns in which to place the group Q number. There are 4 choices for this column, and for each of these choices, there are 3 choices of column in which to place the group R number. The group P numbers are then placed in the empty columns.

We then need to place the numbers on the diagonal. There are 2 ways to do this:

> The group P numbers on the diagonal must go in the columns corresponding to the locations of the group Q and R numbers in the 3^{rd} row.
> The group Q and R numbers on the diagonal can be placed in the two remaining columns in 2 ways – either QR or RQ when reading from left to right.

Therefore, there are $12 \times 2 = 24$ ways in which the BINGO card can be completed using this fifth combination.

In total, the number of ways to complete the BINGO card so that the sum of the numbers in the diagonal is 177, and the sum of the numbers in the 3^{rd} row is 177 is $4+4+12+12+24$, which equals 56.