The CENTRE for EDUCATION in MATHEMATICS and COMPUTING

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2016 Canadian Team Mathematics Contest

Individual Problems

1. What is the value of $2^0 + 20^0 + 201^0 + 2016^0$?

2. Zeljko travelled at 30 km/h for 20 minutes and then travelled at 20 km/h for 30 minutes. How far did he travel, in kilometres?

3. The operation $\ominus$ is defined by $a \ominus b = a^b - b^a$. What is the value of $2 \ominus (2 \ominus 5)$?

4. Two fair six-sided dice are tossed and the numbers shown on the top face of each are added together. What is the probability that the resulting sum is less than 10?

5. A palindrome is a positive integer whose digits are the same when read forwards or backwards. For example, 25352 is a five-digit palindrome. What is the largest five-digit palindrome that is a multiple of 15?

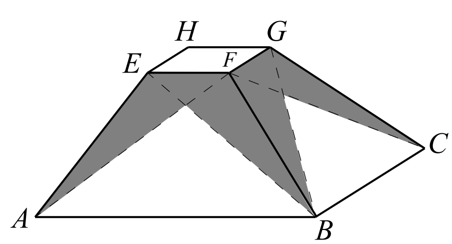
6. On a particular street in Waterloo, there are exactly 14 houses, each numbered with an integer between 500 and 599, inclusive. The 14 house numbers form an arithmetic sequence in which 7 terms are even and 7 terms are odd. One of the houses is numbered 555 and none of the remaining 13 numbers has two equal digits. What is the smallest of the 14 house numbers?

   (An *arithmetic sequence* is a sequence in which each term after the first is obtained from the previous term by adding a constant. For example, 3, 5, 7, 9 is an arithmetic sequence with four terms.)

7. Point Q has coordinates $(a + 1, 4a + 1)$ for some $a > 1$, and lies on the line with equation $y = ax + 3$. If O is the origin $(0, 0)$, determine the coordinates of the points P and R so that $OPQR$ is a square with diagonal OQ.

8. Claudine has p packages containing 19 candies each.
   If Claudine divides all of her candies equally among 7 friends, there are 4 candies left over.
   If Claudine divides all of her candies equally among 11 friends, there is 1 candy left over.
   What is the minimum possible value of p?

9. In the diagram, $ABCDHEFG$ is a truncated square-based pyramid. (Note that D is hidden.) In particular, $ABCD$ is a square with side length 7, $EFGH$ is a square with side length 1, and $AE = BF = CG = DH$. Each of the four trapezoidal faces is partially shaded. On each face, only the region that is below both diagonals is unshaded. If the height of the truncated pyramid is 4, what is the total shaded area?



10. Determine all pairs (x, y) of real numbers that satisfy the following system of inequalities:

$$x^4 + 8x^3y + 16x^2y^2 + 16 \leq 8x^2 + 32xy$$
$$y^4 + 64x^2y^2 + 10y^2 + 25 \leq 16xy^3 + 80xy$$

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Team Problems

1. What is the value of $2 + 0 - 1 \times 6$?

2. The average (mean) of 3, 5, 6, 8, and x is 7. What is the value of x?

3. For any real number x, $\lfloor x \rfloor$ denotes the largest integer less than or equal to x. For example, $\lfloor 4.2 \rfloor = 4$ and $\lfloor -5.4 \rfloor = -6$. What is the value of $\lfloor -2.3 + \lfloor 1.6 \rfloor \rfloor$?

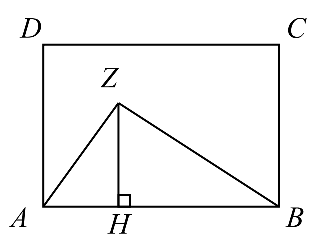
4. A street magician has three cups labelled, in order, A, B, C that he has upside down on his table. He has a sequence of moves that he uses to scramble the three cups: he swaps the first and second, then he swaps the second and third, then he swaps the first and third. If he goes through this sequence of three moves a total of nine times, in what order will the cups be?

5. A parabola has equation $y = ax^2 + bx + c$ and passes through the points $(-3, 50)$, $(-1, 20)$ and $(1, 2)$. Determine the value of $a + b + c$.

6. For some positive integers m and n, $2^m - 2^n = 1792$. Determine the value of $m^2 + n^2$.

7. A two-digit integer between 10 and 99, inclusive, is chosen at random. Each possible integer is equally likely to be chosen. What is the probability that its tens digit is a multiple of its units (ones) digit?

8. Rectangle $ABCD$ has area 2016. Point Z is inside the rectangle and point H is on AB so that ZH is perpendicular to AB. If $ZH : CB = 4 : 7$, what is the area of pentagon $ADCBZ$?



9. Suppose that A and B are digits with

$$\begin{array}{cccc} & A & A & A \\ & A & A & B \\ & A & B & B \\ + & B & B & B \\ \hline 1 & 5 & 0 & 3 \end{array}$$

What is the value of $A^3 + B^2$?

10. Clara takes 2 hours to ride her bicycle from Appsley to Bancroft. The reverse trip takes her 2 hours and 15 minutes. If she travels downhill at 24 km/h, on level road at 16 km/h and uphill at 12 km/h, what is the distance, in kilometres, between the two towns?

11. The first and second terms of a sequence are 4 and 5, respectively. Each term after the second is determined by increasing the previous term by one and dividing the result by the term before that. For example, the third term equals $\dfrac{5+1}{4}$. What is the 1234th term of this sequence?

12. Austin and Joshua play a game. Austin chooses a random number equal to 1, 2, 3, 4, or 5. Joshua then chooses randomly from the remaining four numbers. Joshua's first round score is equal to the product of his number and Austin's number. Austin then chooses randomly from the remaining three numbers, and his first round score is the product of his second number and Joshua's first number. The process is repeated until each of the five numbers has been chosen. The sum of each player's two scores is their final score and the player with the highest final score wins. If Austin chooses 2 to start and Joshua then chooses 3 (making Joshua's first round score 6), what is the probability that Austin will win?

13. A sphere and a cone have the same volume. The area of the lateral surface of the cone is 80π and the total surface area of the cone is 144π. Determine the radius of the sphere.

    (The volume of a sphere with radius r is $\frac{4}{3}\pi r^3$. The volume of a cone with radius r and height h is $\frac{1}{3}\pi r^2 h$. The surface area of a cone consists of two parts: its base and its lateral surface. The lateral surface area of a cone with radius r and slant height s is πrs.)

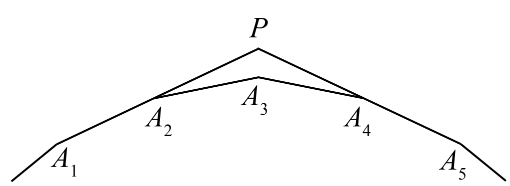
14. Determine the value of the following sum:

$$\log_3(1 - \tfrac{1}{15}) + \log_3(1 - \tfrac{1}{14}) + \log_3(1 - \tfrac{1}{13}) + \cdots + \log_3(1 - \tfrac{1}{8}) + \log_3(1 - \tfrac{1}{7}) + \log_3(1 - \tfrac{1}{6})$$

    (Note that the sum includes a total of 10 terms.)

15. A lock has a combination that is a four-digit positive integer. The first digit is 4 and the four-digit combination is divisible by 45. How many different possible combinations are there?

16. In a regular n-gon, $A_1A_2A_3\cdots A_n$, where $n > 6$, sides A_1A_2 and A_5A_4 are extended to meet at point P. If $\angle A_2PA_4 = 120°$, determine the value of n.



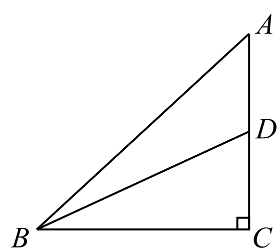
17. There are 21 marbles in a bag. The number of each colour of marble is shown in the following table:

Colour	Number
magenta	1
puce	2
cyan	3
ecru	4
aquamarine	5
lavender	6

For example, the bag contains 4 ecru marbles. Three marbles are randomly drawn from the bag without replacement. The probability that all three of these marbles are the same colour can be written as $\frac{1}{k}$. What is the value of k?

18. For each real number x, $f(x)$ is defined to be the minimum of the values of $2x+3$, $3x-2$ and $25-x$. What is the maximum value of $f(x)$?

19. In the diagram, $\triangle ABC$ is right-angled at C. Point D is on AC so that $\angle ABC = 2\angle DBC$. If $DC = 1$ and $BD = 3$, determine the length of AD.



20. A group of cows and horses are randomly divided into two equal rows. (The animals are well-trained and stand very still.) Each animal in one row is directly opposite an animal in the other row. If 75 of the animals are horses and the number of cows opposite cows is 10 more than the number of horses opposite horses, determine the total number of animals in the group.

21. Three circles each with a radius of 1 are placed such that each circle touches the other two circles, but none of the circles overlap. What is the exact value of the radius of the smallest circle that will enclose all three circles?

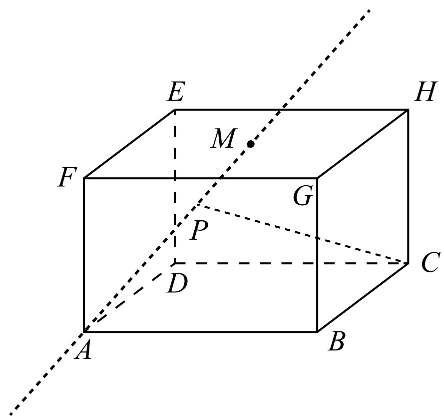
22. In a hospital, there are 7 patients (Doc, Grumpy, Happy, Sleepy, Bashful, Sneezy, and Dopey) who need to be assigned to 3 doctors (Huey, Dewey, and Louie). In how many ways can the patients be assigned to the doctors so that each patient is assigned to exactly one doctor and each doctor is assigned at least one patient?

23. Suppose that a is a positive integer with $a > 1$. Determine a closed form expression, in terms of a, equal to

$$1 + \frac{3}{a} + \frac{5}{a^2} + \frac{7}{a^3} + \cdots$$

(The infinite sum includes exactly the fractions of the form $\frac{2k-1}{a^{k-1}}$ for each positive integer k.)

24. In the diagram, rectangular prism $ABCDEFGH$ has $AB = 2a$, $AD = 2b$, and $AF = 2c$ for some $a, b, c > 0$. Point M is the centre of face $EFGH$ and P is a point on the infinite line passing through A and M. Determine the minimum possible length of line segment CP in terms of a, b, and c.



25. The sequences $t_1, t_2, t_3, \ldots$ and $s_1, s_2, s_3, \ldots$ are defined by

- $t_1 = 1$,
- $t_2 = m$ for some positive integer $m > 0$,
- $s_k = t_1 + t_2 + t_3 + \cdots + t_{k-1} + t_k$ for each $k \geq 1$, and
- $t_n = ns_{n-1}$ for each $n \geq 3$.

There exist positive integers m that end in in exactly four 9s and for which $t_{30} = N!$ for some positive integer N. Determine all of these corresponding values of N.

(If n is a positive integer, the symbol $n!$ (read "n factorial") is used to represent the product of the integers from 1 to n. That is, $n! = n(n-1)(n-2)\cdots(3)(2)(1)$. For example, $5! = 5(4)(3)(2)(1)$ or $5! = 120$.)

0 (a). Evaluate $10 - 2 \times 3$.

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0 (b). Let t be TNYWR.
What is the area of a triangle with base of length $2t$ and height of length $3t + 1$?

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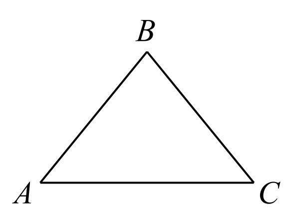
0 (c). Let t be TNYWR.
In the diagram, $\triangle ABC$ is isosceles with $AB = BC$. If $\angle BAC = t^\circ$, what is the measure of $\angle ABC$, in degrees?



1 (a). If $x : 6 = 15 : 10$, what is the value of x?

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1 (b). Let t be TNYWR.
If $\frac{3(x+5)}{4} = t + \frac{3-3x}{2}$, what is the value of x?

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1 (c). Let t be TNYWR.
The y-coordinate of the vertex of the parabola with equation $y = 3x^2 + 6\sqrt{m}x + 36$ is t.
What is the value of m?

2 (a). What is the sum of the x-intercept of the line with equation $20x + 16y - 40 = 0$ and the y-intercept of the line with equation $20x + 16y - 64 = 0$?

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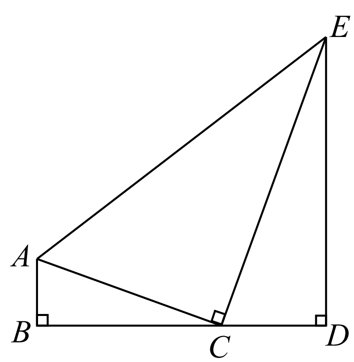
2 (b). Let t be TNYWR.
In the diagram, point C is on BD, $\triangle ABC$ is right-angled at B, $\triangle ACE$ is right-angled at C, and $\triangle CDE$ is right-angled at D. Also, $AB = 2t$, $BD = DE = 9t$, and $BC : CD = 2 : 1$. If the area of $\triangle ACE$ is k, what is the value of $\frac{1}{36}k$?



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2 (c). Let t be TNYWR.
One cylinder has a radius of $\sqrt{2}$ and a height of a. Another cylinder has a radius of $\sqrt{5}$ and a height of b. How many pairs of positive integers (a, b) are there so that the sum of the volumes of the two cylinders is $10\pi t$?

3 (a). Let a be the largest positive integer so that a^3 is less than 999.
Let b be the smallest positive integer so that b^5 is greater than 99.
What is the value of $a - b$?

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3 (b). Let t be TNYWR.
Over the winter, Oscar counted the birds in his backyard. He counted three different types of birds: sparrows, finches and cardinals. Three-fifths of the birds that he counted were sparrows. One-quarter of the birds that he counted were finches. If Oscar counted exactly $10t$ cardinals, how many sparrows did he count?

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3 (c). Let t be TNYWR.
A large theatre has 20 rows of seats. Each row after the first row contains 4 more seats than the previous row. If there are $10t$ seats in total, how many seats are there in the first row?