The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2015 Gauss Contests

(Grades 7 and 8)

Wednesday, May 13, 2015
(in North America and South America)

Thursday, May 14, 2015
(outside of North America and South America)

Solutions

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Grade 7

1. The circle is divided into 4 equal regions. Since 1 of these 4 regions is shaded, then the fraction of the circle that is shaded is $\frac{1}{4}$.

   Answer: (C)

2. Evaluating, $10 \times (5 - 2) = 10 \times 3 = 30$.

   Answer: (D)

3. Reading from the graph, Phil ran 4 km, Tom ran 6 km, Pete ran 2 km, Amal ran 8 km, and Sanjay ran 7 km. Therefore, Pete ran the least distance.

   Answer: (C)

4. The equal-arm balance shows that 2 rectangles have the same mass as 6 circles.
   If we organize these shapes into two equal piles on both sides of the balance, then we see that 1 rectangle has the same mass as 3 circles.

   Answer: (B)

5. Of the possible answers, the length of your thumb is closest to 5 cm.

   Answer: (E)

6. There are 100 centimetres in 1 metre. Therefore, there are $3.5 \times 100 = 350$ cm in 3.5 metres.

   Answer: (A)

7. The length of the side not labelled is equal to the sum of the two horizontal lengths that are labelled, or $2 + 3 = 5$. Thus, the perimeter of the figure shown is $5 + 5 + 2 + 3 + 3 + 2 = 20$.

   Answer: (D)

8. The average (mean) number of points scored per game multiplied by the number of games played is equal to the total number of points scored during the season.
   Therefore, the number of games that Hannah played is equal to the total number of points she scored during the season divided by her average (mean) number of points scored per game, or $312 \div 13 = 24$.

   Answer: (A)

9. The positive divisors of 20 are: $1, 2, 4, 5, 10$, and 20.
   Therefore, the number 20 has exactly 6 positive divisors.

   Answer: (B)

10. Using the digits $4, 7$ and 9 without repeating any digit in a given number, the following 3-digit whole numbers can be formed: $479, 497, 749, 794, 947$, and 974.
    There are exactly 6 different 3-digit whole numbers that can be formed in the manner described.

    Answer: (A)

11. *Solution 1*
    At Gaussville School, 40% or $\frac{40}{100} = \frac{4}{10}$ of the 480 total students voted for math.
    Therefore, the number of students who voted for math is $\frac{4}{10} \times 480 = 4 \times 48 = 192$.

    Solution 2
    At Gaussville School, 40% or 0.4 of the 480 total students voted for math.
    Therefore, the number of students who voted for math is $0.4 \times 480 = 192$.

    Answer: (B)

12. The first fold creates 2 layers of paper. The second fold places 2 sets of 2 layers together, for a total of 4 layers of paper. Similarly, the third fold places 2 sets of 4 layers of paper together, for a total of 8 layers of paper.
That is, each new fold places 2 sets of the previous number of layers together, thereby doubling the previous number of layers.
The results of the first five folds are summarized in the table below.

Number of folds	0	1	2	3	4	5
Number of layers	1	2	4	8	16	32

After the sheet has been folded in half five times, the number of layers in the folded sheet is 32.
ANSWER: (B)

13. *Solution 1*
The multiples of 5 between 1 and 99 are:

$$5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, 95.$$

Of these, only $10, 20, 30, 40, 50, 60, 70, 80$, and 90 are even.
Therefore, there are 9 even whole numbers between 1 and 99 that are multiples of 5.

Solution 2
To create an even multiple of 5, we must multiply 5 by an even whole number (since 5 is odd, multiplying 5 by an odd whole number creates an odd result).
The smallest positive even multiple of 5 is $5 \times 2 = 10$.
The largest even multiple of 5 less than 99 is $5 \times 18 = 90$.
That is, multiplying 5 by each of the even numbers from 2 to 18 results in the only even multiples of 5 between 1 and 99.
Since there are 9 even numbers from 2 to 18 (inclusive), then there are 9 even whole numbers between 1 and 99 that are multiples of 5.
ANSWER: (C)

14. Consider the value of U in the diagram shown.
Since a 3 already occurs in the second row, then U cannot equal 3 (each of the numbers $1, 2, 3$, occur only once in each row).
Since a 1 already occurs in the third column, then U cannot equal 1 (each of the numbers $1, 2, 3$, occur only once in each column).
Since U cannot equal 3 or 1, then $U = 2$.
Therefore, a 2 and a 3 already occur in the second row and so $X = 1$.
At this point, a 2 and a 1 already occur in the third column and so $Y = 3$.
The value of $X + Y = 1 + 3 = 4$.

		1
3	X	U
		Y

ANSWER: (E)

15. The rectangle has area $5 \times 12 = 60\,\text{cm}^2$.
Each of the two congruent unshaded triangles has area $\frac{1}{2} \times 2 \times 5 = 5\,\text{cm}^2$.
The area of the shaded region is equal to the area of the rectangle minus the areas of the two unshaded triangles, which is $60 - 5 - 5 = 50\,\text{cm}^2$.
ANSWER: (E)

16. The total value of one quarter, one dime and one nickel is $25 + 10 + 5 = 40$ ¢.
Since you have equal numbers of quarters, dimes and nickels, you can separate your coins into piles, each containing exactly 1 quarter, 1 dime, and 1 nickel.
Each pile has a value of 40¢, and since $440 \div 40 = 11$, then you must have 11 quarters, 11 dimes and 11 nickels.
Therefore, you have 11 dimes.
Note: You can check that $11 \times (25$ ¢ $+10$ ¢ $+5$ ¢$) = 11 \times 40$ ¢ $= 440$ ¢ $=$ \$4.40, as required.

ANSWER: (B)

17. The original cube (before the corner was cut off) had 12 edges.
Cutting off the corner does not eliminate any of the 12 edges of the original cube.
However, cutting off the corner does add 3 edges that were not present originally, the 3 edges of the new triangular face. Since no edges of the original cube were lost, but 3 new edges were created, then the new solid has $12 + 3 = 15$ edges.

ANSWER: (D)

18. To find the image of PQ, we reflect points P and Q across the x-axis, then join them.
Since P is 3 units above the x-axis, then the reflection of P across the x-axis is 3 units below the x-axis at the same x-coordinate.
That is, point T is the image of P after it is reflected across the x-axis.
Similarly, after a reflection across the x-axis, the image of point Q will be 6 units below the x-axis but have the same x-coordinate as Q.
That is, point U is the image of Q after it is reflected across the x-axis.
Therefore, the line segment TU is the image of PQ after it is reflected across the x-axis.

ANSWER: (B)

19. Since the number of digits that repeat is 6, then the digits 142857 begin to repeat again after 120 digits (since $120 = 6 \times 20$).
That is, the 121^{st} digit is a 1, the 122^{nd} digit is a 4, and the 123^{rd} digit is a 2.

ANSWER: (C)

20. Since the sum of the measures of the three angles in any triangle is $180°$, then the sum of the measures of the two unknown angles in the given triangle is $180° - 45° = 135°$.
The measures of the two unknown angles are in the ratio $4 : 5$, and so one of the two angle measures is $\frac{5}{4+5} = \frac{5}{9}$ of the sum of the two angles, while the other angle measures $\frac{4}{4+5} = \frac{4}{9}$ of the sum of the two angles.
That is, the larger of the two unknown angles measures $\frac{5}{9} \times 135° = 75°$, and the smaller of the unknown angles measures $\frac{4}{9} \times 135° = 60°$.
We may check that $60° + 75° + 45° = 180°$.
The largest angle in the triangle measures $75°$.

ANSWER: (C)

21. We begin by choosing the largest number in each row, $5, 10, 15, 20, 25$, and calling this list L.
The sum of the five numbers in L is $5 + 10 + 15 + 20 + 25 = 75$ and this sum satisfies the condition that no two numbers come from the same row.
However, the numbers in L are taken from columns 1 and 5 only, and the numbers must be chosen so that no two come from the same column.
Thus, the largest of the five answers given, 75, is not possible.
Note: In assuring that we take one number from each row, this choice of numbers, L, is the only way to obtain a sum of 75 (since we chose the largest number in each row).

Of the five answers given, the next largest answer is 73.
Since L uses the largest number in each row and has a sum of 75, we can obtain a sum of 73 either by replacing one of the numbers in L with a number that is two less, or by replacing two of the numbers in L with numbers that are each one less.
For example, the list $3, 10, 15, 20, 25$ (one change to L) has sum 73 as does the list $4, 9, 15, 20, 25$ (two changes to L).
That is, to obtain a sum of 73 while choosing exactly one number from each row, we must choose at least three of the numbers from L.
However, since two numbers in L lie in column 1 and three numbers from L lie in column 5, it is not possible to choose at least three numbers from L so that no two of the numbers are from the same column.
Any other replacement would give a sum less than 73, which would require the replacement of a number with a larger number in another row to compensate. This is impossible since each row is represented in L by the largest number in the row.
Therefore, it is not possible to obtain a sum of 73.

Of the five answers given, the next largest answer is 71.
By choosing the numbers, $3, 9, 14, 20, 25$ we obtain the sum $3 + 9 + 14 + 20 + 25 = 71$ while satisfying the condition that no two numbers come from the same row and no two numbers come from the same column.
Thus, 71 is the largest possible sum that satisfies the given conditions.
Note: There are other choices of five numbers which also give a sum of 71 and satisfy the given conditions.

Answer: (C)

22. Since the perimeter of the square is P and the 4 sides of a square are equal in length, then each side of the square has length $\frac{1}{4}P$. We now work backward to determine the width and length of the rectangle.
The width of the rectangle was doubled to produced the side of the square with length $\frac{1}{4}P$.
Therefore, the width of the rectangle is half of the side length of the square, or $\frac{1}{2} \times \frac{1}{4}P = \frac{1}{8}P$.
The length of the rectangle was halved to produce the side of the square with length $\frac{1}{4}P$.
Therefore, the length of the rectangle is twice the side length of the square, or $2 \times \frac{1}{4}P = \frac{1}{2}P$.
Finally, we determine the perimeter of the rectangle having width $\frac{1}{8}P$ and length $\frac{1}{2}P$, obtaining $2 \times \left(\frac{1}{8}P + \frac{1}{2}P\right) = 2 \times \left(\frac{1}{8}P + \frac{4}{8}P\right) = 2 \times \left(\frac{5}{8}P\right) = \frac{5}{4}P$.

Answer: (D)

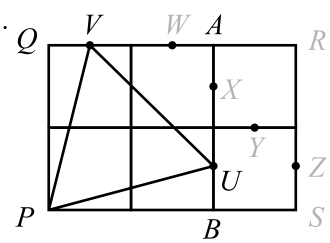
23. *Solution 1*
Every 4-digit palindrome is of the form $abba$, where a is a digit between 1 and 9 inclusive and b is a digit between 0 and 9 inclusive (and b is not necessarily different than a).
Every 5-digit palindrome is of the form $abcba$, where a is a digit between 1 and 9 inclusive, b is a digit between 0 and 9 inclusive (and b is not necessarily different than a), and c is a digit between 0 and 9 inclusive (and c is not necessarily different than a and b).
That is, for every 4-digit palindrome $abba$ there are 10 possible digits c so that $abcba$ is a 5-digit palindrome.
For example if $a = 2$ and $c = 3$, then the 4-digit palindrome 2332 can be used to create the 10 5-digit palindromes: $23032, 23132, 23232, 23332, 23432, 23532, 23632, 23732, 23832, 23932$.
Thus, for every 4-digit palindrome $abba$, there are exactly 10 5-digit palindromes $abcba$ and so the ratio of the number of 4-digit palindromes to the number of 5-digit palindromes is $1 : 10$.

Solution 2

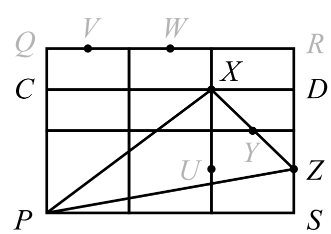
Every 4-digit palindrome is of the form $abba$, where a is a digit between 1 and 9 inclusive and b is a digit between 0 and 9 inclusive (and b is not necessarily different than a).
There are 9 choices for the first digit a and, for each of these choices, there are 10 choices for the second digit b or $9 \times 10 = 90$ choices for the first two digits ab.
Once the first two digits of the 4-digit palindrome are chosen, then the third and fourth digits are also determined (since the third digit must equal the second and the fourth must equal the first).
That is, there are 90 4-digit palindromes.
Every 5-digit palindrome is of the form $defed$, where d is a digit from 1 to 9 inclusive and e is a digit from 0 to 9 inclusive (and e is not necessarily different than d) and f is a digit from 0 to 9 inclusive (and f is not necessarily different than d and e).
There are 9 choices for the first digit d and 10 choices for the second digit e and 10 choices for the third digit f or $9 \times 10 \times 10 = 900$ choices for the first three digits def.
Once the first three digits of the 5-digit palindrome are chosen, then the fourth and fifth digits are also determined (since the fourth digit must equal the second and the fifth must equal the first).
That is, there are 900 5-digit palindromes.
Thus, the ratio of the number of 4-digit palindromes to the number of 5-digit palindromes is $90 : 900$ or $1 : 10$.

ANSWER: (E)

24. We can determine which triangle has the greatest area by using a fixed side length of 4 for each of the identical squares and using this to calculate the unknown areas.

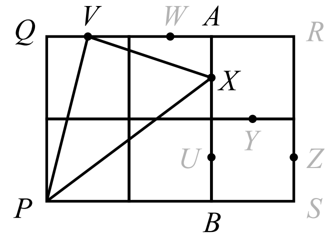


We begin by constructing $\triangle PVU$ and noticing that it is contained within square $QABP$, as shown. The area of $\triangle PVU$ is determined by subtracting the areas of triangles PQV, VAU and PBU from the area of square $QABP$.
Since $QA = 8$ and $AB = 8$, then the area of square $QABP$ is $8 \times 8 = 64$.
Since $PQ = 8$ and $QV = 2$, then the area of $\triangle PQV$ is $\frac{1}{2} \times 8 \times 2 = 8$.
Since $VA = 6$ and $AU = 6$, then the area of $\triangle VAU$ is $\frac{1}{2} \times 6 \times 6 = 18$.
Since $PB = 8$ and $UB = 2$, then the area of $\triangle PBU$ is $\frac{1}{2} \times 8 \times 2 = 8$.
Therefore, the area of $\triangle PVU$ is $64 - 8 - 18 - 8 = 30$.

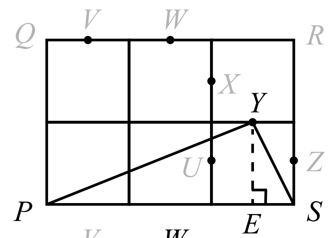


Next, we construct $\triangle PXZ$ and then construct rectangle $CDSP$ by drawing CD parallel to PS through X. Further, X is the midpoint of the side of a square and so C and D are also midpoints of the sides of their respective squares.

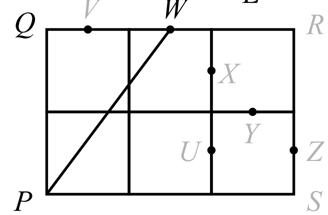
The area of $\triangle PXZ$ is determined by subtracting the areas of triangles PCX, XDZ and PSZ from the area of rectangle $CDSP$.
Since $CD = 12$ and $DS = 6$, then the area of rectangle $CDSP$ is $12 \times 6 = 72$.
Since $PC = 6$ and $CX = 8$, then the area of $\triangle PCX$ is $\frac{1}{2} \times 6 \times 8 = 24$.
Since $XD = 4$ and $DZ = 4$, then the area of $\triangle XDZ$ is $\frac{1}{2} \times 4 \times 4 = 8$.
Since $PS = 12$ and $ZS = 2$, then the area of $\triangle PSZ$ is $\frac{1}{2} \times 12 \times 2 = 12$.
Therefore, the area of $\triangle PXZ$ is $72 - 24 - 8 - 12 = 28$.



Construct $\triangle PVX$ and notice that it is contained within square $QABP$, as shown. The area of $\triangle PVX$ is determined by subtracting the areas of triangles PQV, VAX and PBX from the area of square $QABP$.
As we previously determined, the area of square $QABP$ is 64 and the area of $\triangle PQV$ is 8.
Since $VA = 6$ and $AX = 2$, then the area of $\triangle VAX$ is $\frac{1}{2} \times 6 \times 2 = 6$.
Since $PB = 8$ and $XB = 6$, then the area of $\triangle PBX$ is $\frac{1}{2} \times 8 \times 6 = 24$.
Therefore, the area of $\triangle PVX$ is $64 - 8 - 6 - 24 = 26$.



Construct $\triangle PYS$ and the perpendicular from Y to E on PS, as shown. Since $PS = 12$ and $YE = 4$ (YE is parallel to RS and thus equal in length to the side of the square), then the area of $\triangle PYS$ is $\frac{1}{2} \times 12 \times 4 = 24$.



Construct $\triangle PQW$, as shown. Since $PQ = 8$ and $QW = 6$, then the area of $\triangle PQW$ is $\frac{1}{2} \times 8 \times 6 = 24$.

The areas of the 5 triangles are $30, 28, 26, 24$, and 24. The triangle with greatest area, 30, is $\triangle PVU$.

Answer: (A)

25. All 2-digit prime numbers are odd numbers, so to create a reversal pair, both digits of each prime must be odd (so that both the original number and its reversal are odd numbers).
We also note that the digit 5 cannot appear in either prime number of the reversal pair since any 2-digit number ending in 5 is not prime.
Combining these two facts together leaves only the following list of prime numbers from which to search for reversal pairs: $11, 13, 17, 19, 31, 37, 71, 73, 79$, and 97.
This allows us to determine that the only reversal pairs are: 13 and 31, 17 and 71, 37 and 73, and 79 and 97.
(Note that the reversal of 11 does not produce a different prime number and the reversal of 19 is 91, which is not prime since $7 \times 13 = 91$.)

Given a reversal pair, we must determine the prime numbers (different than each prime of the reversal pair) whose product with the reversal pair is a positive integer less than 10 000.

The product of the reversal pair 79 and 97 is $79 \times 97 = 7663$.
Since the smallest prime number is 2 and $2 \times 7663 = 15\,326$, which is greater than 10 000, then the reversal pair 79 and 97 gives no possibilities that satisfy the given conditions.

We continue in this way, analyzing the other 3 reversal pairs, and summarize our results in the table below.

Prime Number	Product of the Prime Number with the Reversal Pair			
	13 and 31	17 and 71	37 and 73	79 and 97
2	$2 \times 13 \times 31 = 806$	$2 \times 17 \times 71 = 2414$	$2 \times 37 \times 73 = 5402$	greater than 10 000
3	$3 \times 13 \times 31 = 1209$	$3 \times 17 \times 71 = 3621$	$3 \times 37 \times 73 = 8103$	
5	$5 \times 13 \times 31 = 2015$	$5 \times 17 \times 71 = 6035$	greater than 10 000	
7	$7 \times 13 \times 31 = 2821$	$7 \times 17 \times 71 = 8449$		
11	$11 \times 13 \times 31 = 4433$	greater than 10 000		
13	can't use 13 twice			
17	$17 \times 13 \times 31 = 6851$			
19	$19 \times 13 \times 31 = 7657$			
23	$23 \times 13 \times 31 = 9269$			
29	greater than 10 000			
Total	8	4	2	0

In any column, once we obtain a product that is greater than 10 000, we may stop evaluating subsequent products since they use a larger prime number and thus will exceed the previous product.
In total, there are $8 + 4 + 2 = 14$ positive integers less than 10 000 which have the required property.

Answer: (B)

Grade 8

1. Evaluating, $1000 + 200 - 10 + 1 = 1200 - 10 + 1 = 1190 + 1 = 1191$.

 Answer: (A)

2. Since there are 60 minutes in an hour, then 40 minutes after 10:20 it is 11:00.
 Therefore, 45 minutes after 10:20 it is 11:05.

 Answer: (E)

3. Of the possible answers, the length of your thumb is closest to 5 cm.

 Answer: (E)

4. Reading from the graph, Phil ran 4 km, Tom ran 6 km, Pete ran 2 km, Amal ran 8 km, and Sanjay ran 7 km.
 Ordering these distances from least to greatest, we get Pete ran 2 km, Phil ran 4 km, Tom ran 6 km, Sanjay ran 7 km, and Amal ran 8 km.
 In this ordered list of 5 distances, the median distance is in the middle, the third greatest.
 Therefore, Tom ran the median distance.

 Answer: (B)

5. *Solution 1*
 Since $x + 3 = 10$, then $x = 10 - 3 = 7$.
 When $x = 7$, the value of $5x + 15$ is $5(7) + 15 = 35 + 15 = 50$.

 Solution 2
 When multiplying $x + 3$ by 5, we get $5 \times (x + 3) = 5 \times x + 5 \times 3 = 5x + 15$.
 Since $x + 3 = 10$, then $5 \times (x + 3) = 5 \times 10 = 50$.
 Therefore, $5 \times (x + 3) = 5x + 15 = 50$.

 Answer: (E)

6. The two equal widths, each of length 4, contribute $2 \times 4 = 8$ to the perimeter of the rectangle.
 The two lengths contribute the remaining $42 - 8 = 34$ to the perimeter.
 Since the two lengths are equal, they each contribute $34 \div 2 = 17$ to the perimeter.
 Therefore, the length of the rectangle is 17.

 Answer: (B)

7. To begin, there are 4 circles and 2 rectangles on the left arm, balanced by 10 circles on the right arm.
 If we remove 4 circles from each side of the equal-arm scale, the scale will remain balanced (since we are removing the same mass from each side).
 That is, the 2 rectangles that will remain on the left arm are equal in mass to the 6 circles that will remain on the right arm.
 Since 2 rectangles are equal in mass to 6 circles, then 1 rectangle has the same mass as 3 circles.

 Answer: (B)

8. *Solution 1*
 A 5% increase in 160 is equal to $\frac{5}{100} \times 160$ or $0.05 \times 160 = 8$.
 Therefore, Aidan's height increased by 8 cm over the summer.
 His height at the end of the summer was $160 + 8 = 168$ cm.

 Solution 2
 Since Aidan's 160 cm height increased by 5%, then his height at the end of the summer was $(1 + \frac{5}{100}) \times 160 = (1 + 0.05) \times 160 = 1.05 \times 160 = 168$ cm.

 Answer: (A)

9. When $x = 4$ and $y = 2$, $x + y = 4 + 2 = 6$, $xy = 4 \times 2 = 8$, $x - y = 4 - 2 = 2$, $x \div y = 4 \div 2 = 2$, and $y \div x = 2 \div 4 = \frac{1}{2}$.
Therefore, the expression which gives the smallest value when $x = 4$ and $y = 2$ is $y \div x$.

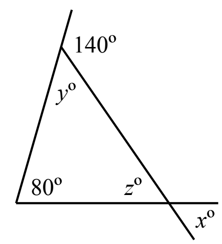
Answer: (E)

10. *Solution 1*
Evaluating using a denominator of 12, $\frac{1}{2} + \frac{1}{4} = \frac{6}{12} + \frac{3}{12} = \frac{9}{12}$ and so the number represented by $\square$ is 9.

Solution 2
Since $\frac{1}{2} + \frac{1}{4} = \frac{2}{4} + \frac{1}{4} = \frac{3}{4}$ and $\frac{3}{4} = \frac{9}{12}$, then $\frac{1}{2} + \frac{2}{4} = \frac{9}{12}$.
The number represented by $\square$ is 9.

Answer: (C)

11. Straight angles measure 180°.
Therefore, $y° + 140° = 180°$, and so $y = 180 - 140 = 40$.
The three interior angles of any triangle add to 180°.
Thus, $40° + 80° + z° = 180°$, and so $z = 180 - 40 - 80 = 60$.
Opposite angles have equal measures.
Since the angle measuring $z°$ is opposite the angle measuring $x°$, then $x = z = 60$.



Answer: (C)

12. Since Zara's bicycle tire has a circumference of 1.5 m, then each full rotation of the tire moves the bike 1.5 m forward.
If Zara travels 900 m on her bike, then her tire will make $900 \div 1.5 = 600$ full rotations.

Answer: (C)

13. To find the image of PQ, we reflect points P and Q across the x-axis, then join them.
Since P is 3 units above the x-axis, then the reflection of P across the x-axis is 3 units below the x-axis at the same x-coordinate.
That is, point T is the image of P after it is reflected across the x-axis.
Similarly, after a reflection across the x-axis, the image of point Q will be 6 units below the x-axis but have the same x-coordinate as Q.
That is, point U is the image of Q after it is reflected across the x-axis.
Therefore, the line segment TU is the image of PQ after it is reflected across the x-axis.

Answer: (B)

14. In the table below, we determine the total value of the three bills that remain in Carolyn's wallet when each of the four bills is removed.

Bill Removed	Sum of the Bills Remaining
\$5	\$10+\$20+\$50=\$80
\$10	\$5+\$20+\$50=\$75
\$20	\$5+\$10+\$50=\$65
\$50	\$5+\$10+\$20=\$35

It is equally likely that any one of the four bills is removed from the wallet and therefore any of the four sums of the bills remaining in the wallet is equally likely.
Of the four possible sums, \$80, \$75, \$65, and \$35, two are greater than \$70.
Therefore, the probability that the total value of the three bills left in Carolyn's wallet is greater than \$70, is $\frac{2}{4}$ or 0.5.

Answer: (A)

15. In the table below, we list the mass of each dog at the end of each month.

Month	0	1	2	3	4	5	6	7	8	9	10	11	12
Walter's mass (in kg)	12	14	16	18	20	22	24	26	28	30	32	34	36
Stanley's mass (in kg)	6	8.5	11	13.5	16	18.5	21	23.5	26	28.5	31	33.5	36

After 12 months have passed, Stanley's mass is 36 kg and is equal to Walter's mass.
(Note that since Stanley's mass is increasing at a greater rate than Walter's each month, this is the only time that the two dogs will have the same mass.)

Answer: (D)

16. First, we must determine the perimeter of the given triangle.
Let the unknown side length measure x cm.
Since the triangle is a right-angled triangle, then by the Pythagorean Theorem we get $x^2 = 8^2 + 6^2$ or $x^2 = 64 + 36 = 100$ and so $x = \sqrt{100} = 10$ (since $x > 0$).
Therefore the perimeter of the triangle is $10+8+6 = 24$ cm and so the perimeter of the square is also 24 cm.
Since the 4 sides of the square are equal in length, then each measures $\frac{24}{4} = 6$ cm.
Thus, the area of the square is $6 \times 6 = 36$ cm^2.

Answer: (D)

17. Since the number of digits that repeat is 6, then the digits 142857 begin to repeat again after 120 digits (since $120 = 6 \times 20$).
That is, the 121^{st} digit is a 1, the 122^{nd} digit is a 4, and the 123^{rd} digit is a 2.

Answer: (C)

18. Using the definition of Δ, we see that $p\Delta 3 = p \times 3 + p + 3 = 3p + p + 3 = 4p + 3$.
Since $p\Delta 3 = 39$, then $4p + 3 = 39$ or $4p = 39 - 3 = 36$ and so $p = \frac{36}{4} = 9$.

Answer: (C)

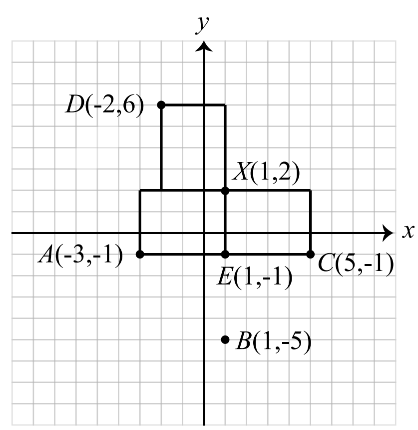
19. *Solution 1*
Originally there are 3 times as many boys as girls, so then for every 3 boys there is 1 girl and $3 + 1 = 4$ children in the room.
That is, the number of boys in the room is $\frac{3}{4}$ of the number of children in the room.
Next we consider each of the 5 possible answers, in turn, to determine which represents the total number of children in the room originally.
If the original number of children in the room is 15 (as in answer (A)), the number of boys is $\frac{3}{4} \times 15 = \frac{45}{4} = 11.25$.
Since it is not possible to have 11.25 boys in the room, then we know that 15 is not the correct answer.
If the original number of children in the room is 20 (as in answer (B)), the number of boys is $\frac{3}{4} \times 20 = \frac{60}{4} = 15$.
If the number of boys in the room was originally 15, then the number of girls was $20 - 15 = 5$.
Next we must check if there will be 5 times as many boys as girls in the room once 4 boys and 4 girls leave the room.
If 4 boys leave the room, there are 11 boys remaining. If 4 girls leave the room, there is 1 girl remaining and since there are not 5 times as many boys as girls, then 20 is not the correct answer.
If the original number of children in the room is 24 (as in answer (C)), the number of boys is $\frac{3}{4} \times 24 = \frac{72}{4} = 18$.
If the number of boys in the room was originally 18, then the number of girls was $24 - 18 = 6$.

If 4 boys leave the room, there are 14 boys left and if 4 girls leave the room, then there are 2 girls left.
Since there are not 5 times as many boys as girls, then 24 is not the correct answer.
If the original number of children in the room is 32 (as in answer (D)), the number of boys is $\frac{3}{4} \times 32 = \frac{96}{4} = 24$.
If the number of boys in the room was originally 24, then the number of girls was $32 - 24 = 8$.
If 4 boys leave the room, there are 20 left and if 4 girls leave the room, then there are 4 left.
Since there are 5 times as many boys as girls, then we know that the original number of children is 32.
(Note: We may check that the final answer, 40, gives 30 boys and 10 girls originally and when 4 boys and 4 girls leave the room there are 26 boys and 6 girls which again does not represent 5 times as many boys as girls.)

Solution 2
Originally there are 3 times as many boys as girls, so if there are x girls in the room, then there are $3x$ boys.
If 4 boys leave the room, there are $3x - 4$ boys remaining.
If 4 girls leave the room, there are $x - 4$ girls remaining.
At this point, there are 5 times as many boys as girls in the room.
That is, $5 \times (x - 4) = 3x - 4$ or $5x - 20 = 3x - 4$ and so $5x - 3x = 20 - 4$ or $2x = 16$ and so $x = 8$.
Therefore, the original number of girls in the room is 8 and the original number of boys is $3 \times 8 = 24$.
The original number of students in the room is $24 + 8 = 32$.

Answer: (D)

20. *Solution 1*
Call the given vertex of the rectangle $(1, 2)$ point X and name each of the 5 answers to match the letters A through E, as shown.



Point $E(1, -1)$ is 3 units below point X (since their x-coordinates are equal and their y-coordinates differ by 3). Thus, $E(1, -1)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and E would be adjacent vertices of the rectangle).

Point $C(5, -1)$ is 3 units below and 4 units right of point X (since their y-coordinates differ by 3 and their x-coordinates differ by 4). Thus, $C(5, -1)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and C would be opposite vertices of the rectangle).

Point $A(-3, -1)$ is 3 units below and 4 units left of point X (since their y-coordinates differ by 3 and their x-coordinates differ by 4). Thus, $A(-3, -1)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and A would be opposite vertices of the rectangle).

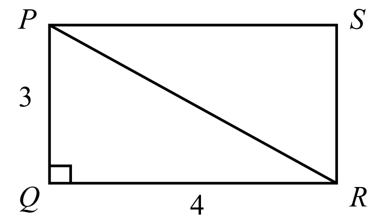
Point $D(-2, 6)$ is 4 units above and 3 units left of point X (since their y-coordinates differ by 4 and their x-coordinates differ by 3). Thus, $D(-2, 6)$ could be the coordinates of a vertex of a 3 by 4 rectangle having vertex X (X and D would be opposite vertices of the rectangle).

The only point remaining is $B(1, -5)$ and since it is possible for each of the other 4 answers to

be one of the other vertices of the rectangle, then it must be $(1, -5)$ that can not be.
Point $B(1, -5)$ is 7 units below point X (since their y-coordinates differ by 7).
How might we show that no two vertices of a 3 by 4 rectangle are 7 units apart?
(See Solution 2).

Solution 2

The distance between any two adjacent vertices of a 3 by 4 rectangle $PQRS$ is either 3 or 4 units (such as P and Q or Q and R, as shown).



The distance between any two opposite vertices of a rectangle (such as P and R) can be found using the Pythagorean Theorem.
In the right-angled triangle PQR, we get $PR^2 = 3^2 + 4^4$ or $PR^2 = 9 + 16 = 25$ and so $PR = \sqrt{25} = 5$ (since $PR > 0$).
That is, the greatest distance between any two vertices of a 3 by 4 rectangle is 5 units.
As shown and explained in Solution 1, the distance between $X(1, 2)$ and $B(1, -5)$ is 7 units.
Therefore $(1, -5)$ could not be the coordinates of one of the other vertices of a 3 by 4 rectangle having vertex $X(1, 2)$.

Answer: (B)

21. In square $PQRS$, $PS = SR$ and since M and N are midpoints of these sides having equal length, then $MS = SN$.
The area of $\triangle SMN$ is $\frac{1}{2} \times MS \times SN$.
Since this area equals 18, then $\frac{1}{2} \times MS \times SN = 18$ or $MS \times SN = 36$ and so $MS = SN = 6$ (since they are equal in length).
The side of the square, PS, is equal in length to $PM + MS = 6 + 6 = 12$ (since M is the midpoint of PS) and so $PS = SR = RQ = QP = 12$.
The area of $\triangle QMN$ is equal to the area of square $PQRS$ minus the combined areas of the three right-angled triangles, $\triangle SMN$, $\triangle NRQ$ and $\triangle QPM$.
Square $PQRS$ has area $PS \times SR = 12 \times 12 = 144$.
$\triangle SMN$ has area 18, as was given in the question.
$\triangle NRQ$ has area $\frac{1}{2} \times QR \times RN = \frac{1}{2} \times 12 \times 6 = 36$ (since $SN = RN = 6$).
$\triangle QPM$ has area $\frac{1}{2} \times QP \times PM = \frac{1}{2} \times 12 \times 6 = 36$.
Thus the area of $\triangle QMN$ is $144 - 18 - 36 - 36 = 54$.

Answer: (E)

22. Let the number of adult tickets sold be a.
Since the price for each adult ticket is \$12, then the revenue from all adult tickets sold (in dollars) is $12 \times a$ or $12a$.
Since the number of child tickets sold is equal to the number of adult tickets sold, we can let the number of child tickets sold be a, and the total revenue from all \$6 child tickets be $6a$ (in dollars).
In dollars, the combined revenue of all adult tickets and all child tickets is $12a + 6a = 18a$.

Since the total number of tickets sold is 120, and a adult tickets were sold and a child tickets were sold, then the remaining $120 - 2a$ tickets were sold to seniors.
Since the price for each senior ticket is \$10, then the revenue from all senior tickets sold (in dollars) is $10 \times (120 - 2a)$.
Thus the combined revenue from all ticket sales is $10 \times (120 - 2a) + 18a$, dollars.

The total revenue from the ticket sales was \$1100 and so $10 \times (120 - 2a) + 18a = 1100$.
Solving this equation, we get $10 \times 120 - 10 \times 2a + 18a = 1100$ or $1200 - 20a + 18a = 1100$ or $1200 - 2a = 1100$ and so $2a = 100$ or $a = 50$.
Therefore, the number of senior tickets sold for the concert was
$120 - 2a = 120 - 2(50) = 120 - 100 = 20$.

We may check that the number of tickets sold to each of the three groups gives the correct total revenue.
Since the number of adult tickets sold was equal to the number of child tickets sold which was equal to a, then 50 of each were sold.
The revenue from 50 adult tickets is $50 \times \$12 = \$600$.
The revenue from 50 child tickets is $50 \times \$6 = \$300$.
The revenue from 20 senior tickets is $20 \times \$10 = \$200$.
The total revenue from all tickets sold was $\$600 + \$300 + \$200 = \$1100$, as required.

Answer: (B)

23. The list of integers $4, 4, x, y, 13$ has been arranged from least to greatest, and so $4 \le x$ and $x \le y$ and $y \le 13$.
The sum of the 5 integers is $4 + 4 + x + y + 13 = 21 + x + y$ and so the average is $\dfrac{21 + x + y}{5}$.
Since this average is a whole number, then $21 + x + y$ must be divisible by 5 (that is, $21 + x + y$ is a multiple of 5).
How small and how large can the sum $21 + x + y$ be?
We know that $4 \le x$ and $x \le y$, so the smallest that $x + y$ can be is $4 + 4 = 8$.
Since $x + y$ is at least 8, then $21 + x + y$ is at least $21 + 8 = 29$.
Using the fact that $x \le y$ and $y \le 13$, the largest that $x + y$ can be is $13 + 13 = 26$.
Since $x + y$ is at most 26, then $21 + x + y$ is at most $21 + 26 = 47$.
The multiples of 5 between 29 and 47 are $30, 35, 40$, and 45.
When $21 + x + y = 30$, we get $x + y = 30 - 21 = 9$.
The only ordered pair (x, y) such that $4 \le x$ and $x \le y$ and $y \le 13$, and $x + y = 9$ is $(x, y) = (4, 5)$.
Continuing in this way, we determine all possible values of x and y that satisfy the given conditions in the table below.

Value of $21 + x + y$	Value of $x + y$	Ordered Pairs (x, y) with $4 \le x$ and $x \le y$ and $y \le 13$
30	$30 - 21 = 9$	$(4, 5)$
35	$35 - 21 = 14$	$(4, 10), (5, 9), (6, 8), (7, 7)$
40	$40 - 21 = 19$	$(6, 13), (7, 12), (8, 11), (9, 10)$
45	$45 - 21 = 24$	$(11, 13), (12, 12)$

The number of ordered pairs (x, y) such that the average of the 5 integers $4, 4, x, y, 13$ is itself an integer is 11.

Answer: (E)

24. The two joggers meet every 36 seconds.
Therefore, the combined distance that the two joggers run every 36 seconds is equal to the total distance around one lap of the oval track, which is constant.
Thus the greater the first jogger's constant speed, the greater the distance that they run every 36 seconds, meaning the second jogger runs less distance in the same time (their combined

distance is constant) and hence the smaller the second jogger's constant speed.
Conversely, the slower the first jogger's constant speed, the less distance that they run every 36 seconds, meaning the second jogger must run a greater distance in this same time and hence the greater the second jogger's constant speed.

This tells us that if the first jogger completes one lap of the track as fast as possible, which is in 80 seconds, then the second jogger's time to complete one lap of the track is as slow as possible.
We will call this time t_{max}, the maximum possible time that it takes the second jogger to complete one lap of the track.
Similarly, if the first jogger completes one lap of the track as slowly as possible, which is in 100 seconds, then the second jogger's time to complete one lap of the track is as fast as possible.
We will call this time t_{min}, the minimum possible time that it takes the second jogger to complete one lap of the track.

Finding the value of t_{max}

Recall that t_{max} is the time it takes the second jogger to complete one lap when the first jogger completes one lap of the track in 80 seconds.
If the first jogger can complete one lap of the track in 80 seconds, then in 36 seconds of running, the first jogger will complete $\frac{36}{80} = \frac{9}{20}$ of a complete lap of the track.
In this same 36 seconds, the two joggers combined distance running is 1 lap, and so the second jogger runs $1 - \frac{9}{20} = \frac{11}{20}$ of a complete lap.
If the second jogger runs $\frac{11}{20}$ of a complete lap in 36 seconds, then the second jogger runs $\frac{20}{11} \times \frac{11}{20} = 1$ complete lap in $\frac{20}{11} \times 36 = \frac{720}{11}$ seconds. Thus, $t_{max} = \frac{720}{11} = 65.\overline{45}$ seconds.

Finding the value of t_{min}

Recall that t_{min} is the time it takes the second jogger to complete one lap when the first jogger completes one lap of the track in 100 seconds.
If the first jogger can complete one lap of the track in 100 seconds, then in 36 seconds of running, the first jogger will complete $\frac{36}{100} = \frac{9}{25}$ of a complete lap of the track.
In this same 36 seconds, the two joggers combined distance running is 1 lap, and so the second jogger runs $1 - \frac{9}{25} = \frac{16}{25}$ of a complete lap.
If the second jogger runs $\frac{16}{25}$ of a complete lap in 36 seconds, then the second jogger runs $\frac{25}{16} \times \frac{16}{25} = 1$ complete lap in $\frac{25}{16} \times 36 = \frac{900}{16}$ seconds. Thus, $t_{min} = \frac{900}{16} = 56.25$ seconds.

Determining the product of the smallest and largest integer values of t

Since the second jogger completes 1 lap of the track in at most $65.\overline{45}$ seconds, then the largest possible integer value of t is 65 seconds.
The second jogger completes 1 lap of the track in at least 56.25 seconds, so then the smallest possible integer value of t is 57 seconds.
Finally, the product of the smallest and largest integer values of t is $57 \times 65 = 3705$.

ANSWER: (A)

25. Let the alternating sum of the digits be S.
If the 7-digit integer is $abcdefg$, then $S = a - b + c - d + e - f + g$.
This sum can be grouped into the digits which contribute positively to the sum, and those which contribute negatively to the sum.
Rewriting the sum in this way, we get $S = (a + c + e + g) - (b + d + f)$.
Taking the 4 digits which contribute positively to S (there are always 4), we let $P = a+c+e+g$.
Similarly, taking the 3 digits which contribute negatively to S (there are always 3), we let $N = b + d + f$.
Thus, it follows that $S = (a + c + e + g) - (b + d + f) = P - N$.

We determine the largest possible value of S by choosing the 4 largest integers, $4, 5, 6, 7$ (in any order), to make up P, and choosing the 3 smallest integers, $1, 2, 3$ (in any order), to make up N.
That is, the largest possible alternating sum is $S = (4 + 5 + 6 + 7) - (1 + 2 + 3) = 16$.
We determine the smallest possible value of S by choosing the 4 smallest integers, $1, 2, 3, 4$ (in any order), to make up P, and choosing the 3 largest integers, $5, 6, 7$ (in any order), to make up N.
That is, the smallest possible alternating sum is $S = (1 + 2 + 3 + 4) - (5 + 6 + 7) = -8$.
Since S must be divisible by 11 (with $S \geq -8$ and $S \leq 16$), then either $S = 11$ or $S = 0$.
The sum of the first 7 positive integers is $1 + 2 + 3 + 4 + 5 + 6 + 7 = 28$, and since each of these 7 integers must contribute to either P or to N, then $P + N = 28$.

Case 1: The alternating sum of the digits is 11, or $S = 11$

If $S = 11$, then $S = P - N = 11$. If $P - N$ is 11 (an odd number), then either P is an even number and N is odd, or the opposite is true (they can't both be odd and they can't both be even).
That is, the difference between two integers is odd only if one of the integers is even and the other is odd (we say that P and N have *different parity*).
However, if one of P or N is even and the other is odd, then their sum $P + N$ is also odd.
But we know that $P + N = 28$, an even number.
Therefore, it is not possible that $S = 11$.
There are no 7-digit integers formed from the integers 1 through 7 that have an alternating digit sum of 11 and are divisible by 11.

Case 2: The alternating sum of the digits is 0, or $S = 0$

If $S = 0$, then $S = P - N = 0$ and so $P = N$.
Since $P + N = 28$, then $P = N = 14$.
We find all groups of 3 digits, chosen from the digits 1 to 7, such that their sum $N = 14$.
There are exactly 4 possibilities: $(7, 6, 1), (7, 5, 2), (7, 4, 3)$, and $(6, 5, 3)$.
In each of these 4 cases, the digits from 1 to 7 that were not chosen, $(2, 3, 4, 5), (1, 3, 4, 6), (1, 2, 5, 6)$, and $(1, 2, 4, 7)$, respectively, represent the 4 digits whose sum is $P = 14$.
We summarize this in the table below.

4 digits whose sum is $P = 14$	3 digits whose sum is $N = 14$	2 examples of 7-digit integers created from these
$2, 3, 4, 5$	$7, 6, 1$	$2736415, 3126475$
$1, 3, 4, 6$	$7, 5, 2$	$1735426, 6745321$
$1, 2, 5, 6$	$7, 4, 3$	$2714536, 5763241$
$1, 2, 4, 7$	$6, 5, 3$	$4615237, 7645231$

Consider the first row of numbers in this table above.
Each arrangement of the 4 digits $2, 3, 4, 5$ combined with each arrangement of the 3 digits $7, 6, 1$ (in the required way) gives a new 7-digit integer whose alternating digit sum is 0.
Two such arrangements are shown (you may check that $S = 0$ for each).
Since there are $4 \times 3 \times 2 \times 1 = 24$ ways to arrange the 4 digits (4 choices for the first digit, 3 choices for the second, 2 choices for the third and 1 choice for the last digit), and $3 \times 2 \times 1 = 6$ ways to arrange the 3 digits, then there are $24 \times 6 = 144$ ways to arrange the 4 digits and the 3 digits.
Each of these 144 arrangements is different from the others, and since $P = N = 14$ for each, then $S = P - N = 0$ and so each of the 144 7-digit numbers is divisible by 11.
Similarly, there are also 144 arrangements that can be formed with each of the other 3 groups of integers that are shown in the final 3 rows of the table.
That is, there are a total of $144 \times 4 = 576$ 7-digit integers (formed from the integers 1 through 7) which are divisible by 11.

The total number of 7-digit integers that can be formed from the integers 1 through 7 is equal to the total number of arrangements of the integers 1 through 7, or $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.
Therefore, when the digits 1 through 7 are each used to form a random 7-digit integer, the probability that the number formed is divisible by 11 is $\frac{576}{5040} = \frac{4}{35}$.

Answer: (E)