The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
cemc.uwaterloo.ca

2015 Galois Contest

Thursday, April 16, 2015
(in North America and South America)

Friday, April 17, 2015
(outside of North America and South America)

Solutions

1. (a) The x-intercept of a line occurs at the point whose y-coordinate is 0.
Substituting $y = 0$ into the equation of line 1, we get $0 = 2x + 6$ or $2x = -6$ and so $x = -3$.
Line 1 has x-intercept -3. (Point P has coordinates $(-3, 0)$.)

(b) *Solution 1*
The equation of line 2 with slope -3 and y-intercept b is $y = -3x + b$.
Line 2 passes through $Q(3, 12)$ and so $x = 3$ and $y = 12$ satisfies the equation of line 2.
Substituting, we get $12 = -3(3) + b$ or $12 = -9 + b$ and so $b = 21$.
The equation of line 2 is $y = -3x + 21$.

Solution 2
A line with slope m and passing through the point (x_1, y_1) has equation $y - y_1 = m(x - x_1)$.
Since line 2 has slope $m = -3$ and passes through $Q(3, 12)$, the slope-point equation of line 2 is $y - 12 = -3(x - 3)$.

(c) We find the coordinates of point R by substituting $y = 0$ into the equation of line 2.
This gives $0 = -3x + 21$ or $3x = 21$ and so $x = 7$.
That is, line 2 has x-intercept 7 (point R has coordinates $(7, 0)$).
If we let the base of $\triangle PQR$ be side PR, then the height of the triangle is the vertical distance from Q to PR.
Since Q has y-coordinate 12, then this height is 12.
The x-coordinate of P is -3 and the x-coordinate of R is 7. (Both y-coordinates are 0.)
Therefore, PR has length $7 - (-3) = 10$.
Finally, the area of $\triangle PQR$ is $\frac{1}{2} \times 10 \times 12 = 60$.

2. (a) The total number of students at School A is the sum of the number of students who received a ride and the number of students who did not, or $330 + 420 = 750$.
Since 330 of 750 students received a ride, the percentage of students at School A who received a ride is $\frac{330}{750} \times 100\% = 0.44 \times 100\% = 44\%$.

(b) *Solution 1*
At school B, 30% of 240 students or $\frac{30}{100} \times 240 = \frac{7200}{100} = 72$ students received a ride.
If 50% or $\frac{1}{2}$ of the students in School B were to receive a ride, then $\frac{1}{2} \times 240 = 120$ students would get a ride.
Therefore, $120 - 72 = 48$ more students needed to receive a ride so that 50% of the students in School B got a ride.

Solution 2
As a percent, the difference between 50% of students receiving a ride and the 30% of students who did receive a ride is 20%.
Therefore, 20% of 240 students or $\frac{20}{100} \times 240 = \frac{4800}{100} = 48$ additional students would need to receive a ride so that 50% of students in School B got a ride.

(c) *Solution 1*
At school C, 45% of 200 students or $\frac{45}{100} \times 200 = \frac{9000}{100} = 90$ students received a ride.
At school D, $x\%$ of 300 students or $\frac{x}{100} \times 300 = \frac{300x}{100} = 3x$ students received a ride.
The total number of students at School C and School D is $200 + 300 = 500$.
The total number of students receiving a ride at School C and School D is $90 + 3x$.
Since 57.6% of the combined group of students from the two schools received a ride, then $\frac{90+3x}{500} = \frac{57.6}{100}$.
Multiplying both sides of this equation by 500, we get $90 + 3x = 57.6 \times 5$ or $90 + 3x = 288$ or $3x = 198$ and so $x = 66$.

Solution 2
At school C, 45% of 200 students or $\frac{45}{100} \times 200 = \frac{9000}{100} = 90$ students received a ride.
The total number of students at School C and School D is $200 + 300 = 500$.
Since 57.6% of the combined group of students from the two schools received a ride, then $\frac{57.6}{100} \times 500 = \frac{28800}{100} = 288$ students received a ride.
Out of the 288 students who received a ride, 90 students were from School C and so the remaining $288 - 90 = 198$ students were from School D.
Since there are 300 students at School D, the percentage of students receiving a ride is $\frac{198}{300} \times 100\% = 0.66 \times 100\% = 66\%$.
Therefore, the value of x is 66.

(d) At school E, $n\%$ of 200 students or $\frac{n}{100} \times 200 = \frac{200n}{100} = 2n$ students received a ride.
At school F, $2n\%$ of 250 students or $\frac{2n}{100} \times 250 = \frac{500n}{100} = 5n$ students received a ride.
The total number of students at School E and School F is $200 + 250 = 450$.
The total number of students receiving a ride at School E and School F is $2n + 5n = 7n$.
Between 55% and 60% of the 450 students from the two schools received a ride.
Since 55% of 450 is 247.5 and 60% of 450 is 270, then $7n > 247.5$ and $7n < 270$.
Solving $7n > 247.5$ we get $n > 35.35$, after rounding, and $7n < 270$ gives $n < 38.57$, after rounding.
Since n is a positive integer and $n > 35.35$ and $n < 38.57$, then the possible values of n are $36, 37$ and 38.

3. (a) Since 5 is an odd integer, then n must be an odd integer for the sum $n + 5$ to be an even integer.
(If n was an even integer, then $n + 5$ would be the sum of an even integer and an odd integer, which is an odd integer.)

(b) We first note that the product of an even integer and any other integers, even or odd, is always an even integer.
Let $N = cd(c + d)$.
If c or d is an even integer (or both c and d are even integers), then N is the product of an even integer and some other integers and thus is even.
The only remaining possibility is that both c and d are odd integers.
If c and d are odd integers, then the sum $c + d$ is an even integer.
In this case, N is again the product of an even integer and some other integers and so it is an even integer.
Therefore, for any integers c and d, $cd(c + d)$ is always an even integer.

(c) Since e and f are positive integers so that $ef = 300$, then we may begin by determining the factor pairs of positive integers whose product is 300.
Written as ordered pairs (x, y) with $x < y$, these are:

$$(1, 300), (2, 150), (3, 100), (4, 75), (5, 60), (6, 50), (10, 30), (12, 25), (15, 20).$$

It is also required that the sum $e + f$ be odd and so exactly one of e or f must be odd.
Therefore, the factor pairs whose sum is odd are:

$$(1, 300), (3, 100), (4, 75), (5, 60), (12, 25), (15, 20).$$

There are 6 ordered pairs (e, f) satisfying the given conditions.

(d) Since both m and n are positive integers, then $2n > 1$ and so $2n + m > m + 1$.
Let $a = m + 1$ and $b = 2n + m$ or $a = 2n + m$ and $b = m + 1$ so that $ab = 9000$.
We must first determine all factor pairs (a, b) of positive integers whose product is 9000.
We begin by considering the parity (whether each is even or odd) of the factors a and b.

Since 2 is even, then $2n$ is even for all positive integers n.
If m is even then $2n + m$ is even since the sum of two even integers is even.
However if m is even, then $m + 1$ is odd since the sum of an even integer and an odd integer is odd.
That is, if m is even, then a is odd and b is even or a is even and b is odd.
We say that the factors a and b have *different parity* since one is even and one is odd.

If m is odd then $2n + m$ is odd. If m is odd then $m + 1$ is even.
That is, if m is odd, then a is even and b is odd or a is odd and b is even and so the factors a and b have different parity for all possible values of m.

Now we are searching for all factor pairs (a, b) of positive integers whose product is 9000 with a and b having different parity.
Written as a product of its prime factors, $9000 = 2^3 \times 3^2 \times 5^3$ and so $ab = 2^3 \times 3^2 \times 5^3$.
Since exactly one of a or b is odd, then one of them does not have a factor of 2 and so the other must have all factors of 2.
That is, either $a = 2^3 r = 8r$ and $b = s$, or $a = r$ and $b = 8s$ for positive integers r and s.
In both cases, $ab = 8rs = 9000$ and so $rs = \frac{9000}{8} = 1125 = 3^2 5^3$.
We now determine all factor pairs (r, s) of positive integers whose product is 1125.
These are $(r, s) = (1, 1125), (3, 375), (5, 225), (9, 125), (15, 75), (25, 45)$.
Therefore $(a, b) = (8r, s) = (8, 1125), (24, 375), (40, 225), (72, 125), (120, 75), (200, 45)$, or $(a, b) = (r, 8s) = (1, 9000), (3, 3000), (5, 1800), (9, 1000), (15, 600), (25, 360)$.
Since $2n + m > m + 1 > 1$, then the pair $(1, 9000)$ is not possible.
This leaves 11 factor pairs (a, b) such that $ab = 9000$ with a and b having different parity.
Each of these 11 factor pairs (a, b) gives an ordered pair (m, n).
To see this, let $m + 1$ equal the smaller of a and b, and let $2n + m$ equal the larger (since $2n + m > m + 1$).
For example when $(a, b) = (8, 1125)$, then $m+1 = 8$ or $m = 7$ and so $2n+m = 2n+7 = 1125$ or $2n = 1118$ or $n = 559$.
That is, the factor pair $(a, b) = (8, 1125)$ corresponds to the ordered pair $(m, n) = (7, 559)$ so that $(m + 1)(2n + m) = 9000$.
Each of the 11 pairs (a, b) gives an ordered pair (m, n) such that $(m + 1)(2n + m) = 9000$.
We determine the corresponding ordered pair (m, n) for each (a, b) in the table below (although this work is not necessary since we were only asked for the number of ordered pairs).

(a,b)	$m+1$	$2n+m$	(m,n)
$(8,1125)$	8	1125	$(7,559)$
$(24,375)$	24	375	$(23,176)$
$(40,225)$	40	225	$(39,93)$
$(72,125)$	72	125	$(71,27)$
$(120,75)$	75	120	$(74,23)$
$(200,45)$	45	200	$(44,78)$

(a,b)	$m+1$	$2n+m$	(m,n)
$(3,3000)$	3	3000	$(2,1499)$
$(5,1800)$	5	1800	$(4,898)$
$(9,1000)$	9	1000	$(8,496)$
$(15,600)$	15	600	$(14,293)$
$(25,360)$	25	360	$(24,168)$

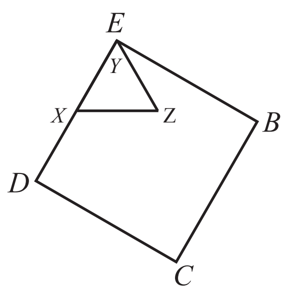
There are 11 ordered pairs (m, n) of positive integers satisfying $(m + 1)(2n + m) = 9000$.

4. (a) Since EXD is a straight line, then $\angle YXE + \angle YXZ = 180^\circ$.
Since $\triangle XYZ$ is an equilateral triangle, then $\angle YXZ = 60^\circ$.
Thus, $\angle YXE = 180^\circ - 60^\circ = 120^\circ$.

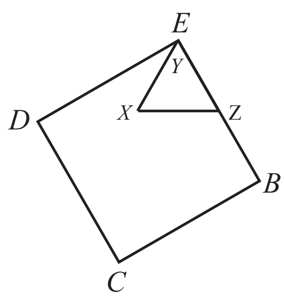
(b) We make a table to track the position of the square as it rotates about the triangle:

After Move #	Coinciding Vertices	Second Vertex of $\triangle XYZ$ on Side of Square	Centre of Rotation of Next Move	Angle of Rotation of Next Move
Initial	D and Z	X on DE	X	120°
1	E and Y	X on DE	Y	30°
2	E and Y	Z on EB	Z	120°
3	B and X	Z on EB	X	30°
4	B and X	Y on BC	Y	120°
5	C and Z	Y on BC	Z	30°
6	C and Z	X on CD	X	120°
7	D and Y	X on CD	Y	30°
8	D and Y	Z on DE	Z	120°

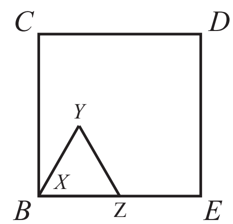
The information in this table comes from the diagrams here:



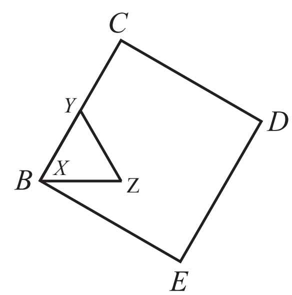
End of
First Move



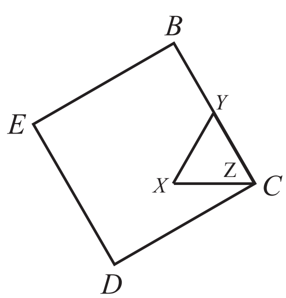
End of
Second Move



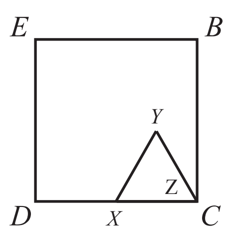
End of
Third Move



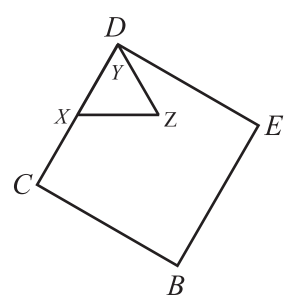
End of
Fourth Move



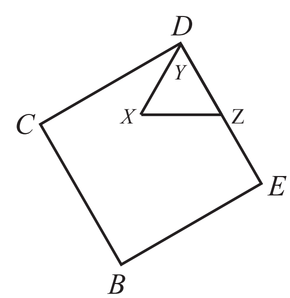
End of
Fifth Move



End of
Sixth Move



End of
Seventh Move



End of
Eighth Move

Each angle of rotation is either $180^\circ - 60^\circ = 120^\circ$ or $90^\circ - 60^\circ = 30^\circ$.
Therefore, D next coincides with a vertex of the triangle after 7 moves.
(We continued the table through the 8th move to be more useful in part (c).)

(c) After 0 moves, the square has D at Z and X on DE and the next rotation is a rotation of 120° about X.
After 8 moves, the square has D at Y and Z on DE and the next rotation is a rotation of 120° about Z.
Therefore, through 8 moves, the net change of position of the square was moving clockwise around 2 sides of the triangle. After these 8 moves, the square is in a similar position, relative to the triangle, to its initial position: D is at a vertex of the triangle and DE lies along a side of the triangle.
Starting from after the 8th move, the square starts in this similar position, and so 8 more moves will take the square again to a similar position. The net change will again be that the square has moved clockwise around 2 sides of the triangle. (We note from part (b) that after 8 moves is the first time (after the original position) that vertex D coincides with a vertex of the triangle at the same relative position.)
Thus, after 16 moves, the square will have D at X and Y on DE and the next rotation is a rotation of 120° about Y. (This is the second time that vertex D coincides with a vertex of the triangle at the same relative position.)
Using similar reasoning, after 24 moves, the square will have D at Z and X on DE and the next rotation is a rotation of 120° about X, which is the original position.
Therefore, we need to determine the total distance travelled by E through these 24 moves.
This total distance equals 3 times the distance travelled by E through the first 8 moves.
This is because the relative position of the square with respect to the triangle (D at a vertex of the triangle, DE along a side of the triangle) is the same after 8 moves as it was after 0 moves, so the relative sequence of rotations undergone by E from after move 8 to after move 16, and from after move 16 to after move 24, will be the same as they were for the first 8 moves.

Since $EBCD$ has side length 2, then $ED = EB = 2$.
Also, the distance from E to the midpoint of ED is 1, as is the distance from E to the midpoint of EB.
Since $EBCD$ has side length 2, then $EC = 2\sqrt{2}$.
Also, the distance from E to the midpoint of each of DC and BC is $\sqrt{2^2 + 1^2} = \sqrt{5}$, by the Pythagorean Theorem.
We make a chart of the rotation undergone by E through each of the first 8 moves:

Move #	Centre of Rotation	Distance of Centre from E	Angle of Rotation of Square
1	X	1	120°
2	Y	0	30°
3	Z	1	120°
4	X	2	30°
5	Y	$\sqrt{5}$	120°
6	Z	$2\sqrt{2}$	30°
7	X	$\sqrt{5}$	120°
8	Y	2	30°

During each rotation, each point on the square (except for the point of the square in contact with the centre of rotation) is rotated about the centre of rotation through the angle of rotation of the square.
Thus, during each rotation, the distance travelled by E is the fraction of the circumference of the whole circle given by the angle of rotation as compared to 360°.

Therefore, the distance travelled by E through the first 8 moves is

$$\frac{120^\circ}{360^\circ}2\pi(1)+\frac{30^\circ}{360^\circ}2\pi(0)+\frac{120^\circ}{360^\circ}2\pi(1)+\frac{30^\circ}{360^\circ}2\pi(2)+$$

$$\frac{120^\circ}{360^\circ}2\pi(\sqrt{5})+\frac{30^\circ}{360^\circ}2\pi(2\sqrt{2})+\frac{120^\circ}{360^\circ}2\pi(\sqrt{5})+\frac{30^\circ}{360^\circ}2\pi(2)$$

which simplifies to

$$\tfrac{2}{3}\pi+0+\tfrac{2}{3}\pi+\tfrac{1}{3}\pi+\tfrac{2}{3}\sqrt{5}\pi+\tfrac{1}{3}\sqrt{2}\pi+\tfrac{2}{3}\sqrt{5}\pi+\tfrac{1}{3}\pi$$

or

$$2\pi+\tfrac{4}{3}\sqrt{5}\pi+\tfrac{1}{3}\sqrt{2}\pi$$

Therefore, when the square returns to its initial position after 24 moves, E has travelled 3 times this far, or a total distance of $6\pi+4\sqrt{5}\pi+\sqrt{2}\pi$.