The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
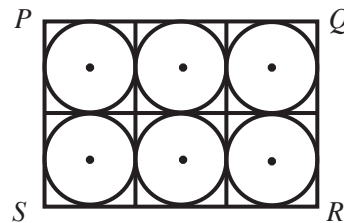### *cemc.uwaterloo.ca*

2015 Fryer Contest

Thursday, April 16, 2015
(in North America and South America)

Friday, April 17, 2015
(outside of North America and South America)

Solutions

1. (a) A Model A cylinder has radius $r = 10$ cm and height $h = 16$ cm and thus has volume
    $V = \pi(10)^2(16) = 1600\pi$ cm^3.

   (b) A Model B cylinder has radius $r = 8$ cm and thus has volume $V = \pi(8)^2(h) = 64\pi h$ cm^3.
    Since the volume of a Model B cylinder is equal to the volume of a Model A cylinder, then
    $64\pi h = 1600\pi$ and so $h = \dfrac{1600\pi}{64\pi} = 25$ cm.
    The height of a Model B cylinder is 25 cm.

   (c) Consider the top-down view of the open Box A shown.
    We label the rectangle as $PQRS$.
    Since each circle has radius 10 cm, then each circle has diameter
    20 cm, and so can be enclosed in a square with side length 20 cm
    whose sides are parallel to the sides of rectangle $PQRS$.
    Each circle touches each of the four sides of its enclosing square.
    Because each of the circles touches one or two sides of rectangle $PQRS$ and each of the
    circles touches two or three of the other circles, then these six squares will fit together
    without overlapping to completely cover rectangle $PQRS$.
    Therefore, $PQ = SR = 3 \times 20 = 60$ cm and $PS = QR = 2 \times 20 = 40$ cm since $PQRS$ is
    three squares wide and two squares tall.
    Finally, since the height of each Model A cylinder is 16 cm, then the height of the box is
    16 cm, and so the volume of Box A is $60 \times 40 \times 16 = 38\,400$ cm^3.

   (d) The volume of Box B is equal to the volume of Box A.
    Intuitively, this makes some sense since the volume of a Model B cylinder is equal to the
    volume of a Model A cylinder and each box is tightly packed with six cylinders.
    We may follow the approach used in part (c) to demonstrate that the volume of Box B is
    also 38 400 cm^3.

    Recall from part (b) that a Model B cylinder has a radius of 8 cm and a height of 25 cm.
    The length of Box B is six times the radius of a Model B cylinder, or $6 \times 8 = 48$ cm.
    The width of Box B is four times the radius of a Model B cylinder, or $4 \times 8 = 32$ cm.
    The height of Box B equals the height of a Model B cylinder, or 25 cm.
    Therefore, the volume of Box B is $48 \times 32 \times 25 = 38\,400$ cm^3 which is equal to the volume
    of Box A.

2. (a) A quarter is worth $0.25 and so 3 quarters are worth $3 \times \$0.25 = \$0.75$.
    A dime is worth $0.10 and so 18 dimes are worth $18 \times \$0.10 = \$1.80$.
    A nickel is worth $0.05 and so 25 nickels are worth $25 \times \$0.05 = \$1.25$.
    The total value of Susan's coins is $\$0.75 + \$1.80 + \$1.25 = \$3.80$.

   (b) *Solution 1*
    Pair each of Allen's nickels with a dime.
    Since there are an equal number of nickels and dimes (and no other coins), then every
    nickel pairs with a dime with no coins left over.
    Each nickel + dime pairing has a value of $\$0.05 + \$0.10 = \$0.15$.
    Since Allen's coins have a total value of $1.50, then he has $\$1.50 \div \$0.15 = 10$ nickel +
    dime pairings.
    Since there is one nickel in each nickel + dime pairing, then Allen must have 10 nickels.

    Solution 2
    Let the number of nickels that Allen has be y.
    Allen has equal numbers of nickels and dimes and so the number of dimes Allen has is

also y.

The value of y nickels is $\$0.05y$.

The value of y dimes is $\$0.10y$.

Since the total value of Allen's coins is $\$1.50$, then $0.05y + 0.10y = 1.50$ or $0.15y = 1.50$ and so $y = \dfrac{1.50}{0.15} = 10$.

Allen has 10 nickels.

(c) A quarter is worth $\$0.25$ and so x quarters are worth $\$0.25x$.

A dime is worth $\$0.10$ and so $2x + 3$ dimes are worth $\$0.10(2x + 3)$.

Since Elise has $\$10.65$ in quarters and dimes, then $0.25x + 0.10(2x + 3) = 10.65$ or $0.25x + 0.20x + 0.30 = 10.65$ or $0.45x = 10.35$ and so $x = \dfrac{10.35}{0.45} = 23$.

3. (a) Using the formula given, the sum of the first 200 positive integers

$$1 + 2 + 3 + \cdots + 198 + 199 + 200 = \frac{200(200 + 1)}{2} = 100(201) = 20\,100.$$

(b) The sum of the first 200 positive integers is equal to the sum of the first 150 positive integers added to the sum of the 50 consecutive integers $151 + 152 + 153 + \cdots + 198 + 199 + 200$. That is,

$$1 + 2 + \cdots + 198 + 199 + 200 = (1 + 2 + \cdots + 148 + 149 + 150) + (151 + 152 + \cdots + 198 + 199 + 200).$$

Therefore, the sum of the 50 consecutive integers $151 + 152 + 153 + \cdots + 198 + 199 + 200$ is equal to the difference between the sum of the first 200 positive integers and the sum of the first 150 positive integers.

From part (a), the sum of the first 200 positive integers is $20\,100$.

Using the formula, the sum of the first 150 positive integers

$$1 + 2 + 3 + \cdots + 148 + 149 + 150 = \frac{150(150 + 1)}{2} = 75(151) = 11\,325.$$

Therefore, the sum of the 50 consecutive integers beginning at 151,

$$\begin{aligned}
151 + 152 + \cdots + 199 + 200 &= (1 + 2 + \cdots + 199 + 200) - (1 + 2 + \cdots + 149 + 150) \\
&= 20\,100 - 11\,325 \\
&= 8775.
\end{aligned}$$

(c) Let the required sum, $1 + 2 + 4 + 5 + 7 + 8 + 10 + 11 + \cdots + 998 + 1000$, be S.

Let the sum of the first 333 positive multiples of 3, $3 + 6 + 9 + 12 + \cdots + 996 + 999$, be T.

The sum of the first 1000 positive integers is equal to $S + T$.

That is,

$$1 + 2 + \cdots + 998 + 999 + 1000 = (1 + 2 + 4 + 5 + 7 + \cdots + 998 + 1000) + (3 + 6 + \cdots + 996 + 999).$$

Therefore, the required sum S is equal to the difference between the sum of the first 1000 positive integers and T.

Using the formula, the sum of the first 1000 positive integers

$$1 + 2 + 3 + \cdots + 998 + 999 + 1000 = \frac{1000(1000 + 1)}{2} = 500(1001) = 500\,500.$$

We may determine the sum T by first removing the common factor 3 from each term and then using the formula.

$$
\begin{aligned}
3 + 6 + 9 + \cdots + 993 + 996 + 999 &= 3(1 + 2 + 3 + \cdots + 331 + 332 + 333) \\
&= 3\left(\frac{333(333 + 1)}{2}\right) \qquad \text{(by the formula)} \\
&= 3(333 \times 167) \\
&= 166\,833.
\end{aligned}
$$

Finally, the required sum S is $500\,500 - 166\,833 = 333\,667$.

4. We number the rows and columns of hexagons as shown:
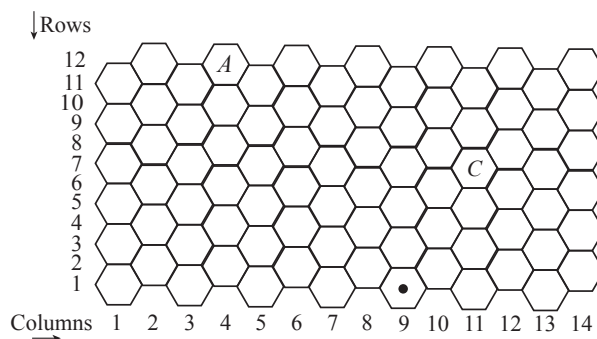
Notice that column 1 includes hexagons only in the odd-numbered rows, column 2 includes hexagons only in the even-numbered rows, and so on.

Therefore, the token starts in the hexagon in row 1, column 9 (we write r1c9), the hexagon labelled A is in row 12, column 4 (r12c4), and the hexagon labelled C is in row 7, column 11 (r7c11).

A ↑ step increases the row number by 2 and does not change the column number.

A ↖ step increases the row number by 1 and decreases the column number by 1.

A ↗ step increases the row number by 1 and increases the column number by 1.

(a) We must move the token from r1c9 to r12c4 in as few steps as possible.

At least 5 ↖ steps are needed in order to move the token 5 columns to the left. (More ↖ steps could be made if they were balanced by ↗ steps.)

One way to move the token from r1c9 to r12c4 is to use 5 ↖ steps (taking the token to r6c4 since each ↖ step moves the token 1 column to the left and 1 row up) and then 3 ↑ steps.

This is 8 steps in total.

Can the token be moved from r1c9 to r12c4 in fewer than 8 steps?

From above, at least 5 ↖ steps are needed. These move the token up 5 of the 11 rows necessary.

To move the token up the remaining 6 rows, at least 3 more steps are needed, since any step moves the token up at most 2 rows.

Therefore, at least 8 steps are required, so the minimum number of steps required is 8.

(b) We must move the token from r1c9 to r12c4 using as many steps as possible.

Each step moves the token at least 1 row higher, so to move $12 - 1 = 11$ rows upwards, at most 11 steps can be used.

The following diagram shows that this can be done using exactly 11 steps:
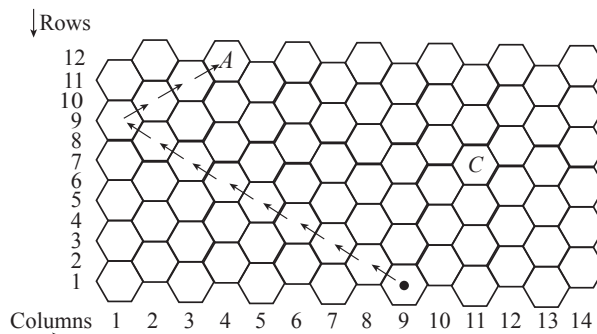
Note that no ↑ steps are used, since each such step moves the token 2 rows higher.
Therefore, the maximum number of steps required is 11.
(Note that there are many other 11 step paths which end at A.)

(c) *Solution 1*

After 1 step, there are three different hexagons that can be reached, each in exactly 1 way.
(These are the hexagons that can be reached using 1 ↖, 1 ↑, or 1 ↗.)
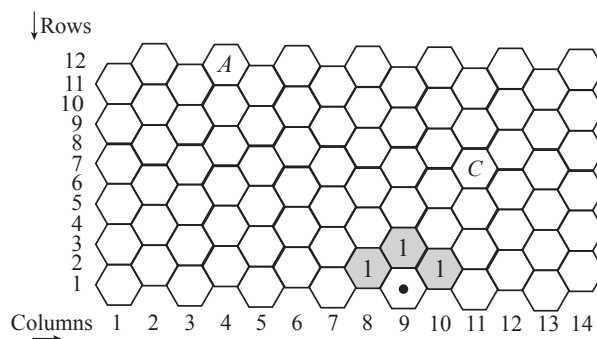

For each positive integer $s \geq 1$, we can determine the hexagons that can be reached using
exactly $s + 1$ steps by starting with all of the hexagons that can be reached in exactly s
steps and moving from each of the three possible directions.

Furthermore, we can determine the number of ways that each of these new hexagons can
be reached in exactly $s + 1$ steps.

The number of ways that a particular hexagon can be reached in exactly $s + 1$ steps is the
sum of the number of ways that each of the three hexagons that are adjacent and below
can be reached in exactly s steps.

This is because every path to a particular hexagon that includes exactly $s + 1$ steps must
be made up of a path of exactly s steps to a hexagon that is adjacent and below followed
by a ↗ step, a ↑ step, or a ↖ step.

For example, in the given diagram, the shaded hexagons can be reached in exactly s steps
in a, b and c ways, and so the hexagon labelled X can be reached in exactly $s + 1$ steps in
$a + b + c$ ways.
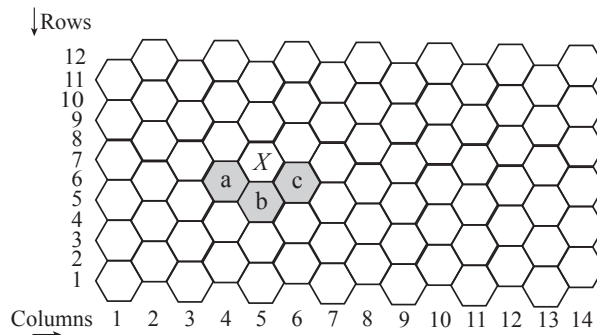
We also note that there are exactly 3^s paths of length s, since there are 3 possibilities for each of the s steps in such a path.

This fact allows us to verify at each step that we have accounted for all possibilities.

Using these facts, we carefully determine the hexagons that can be reached in exactly 2, 3, 4, and 5 steps, and the number of ways that each of these hexagons can be reached. The following diagrams show this information:
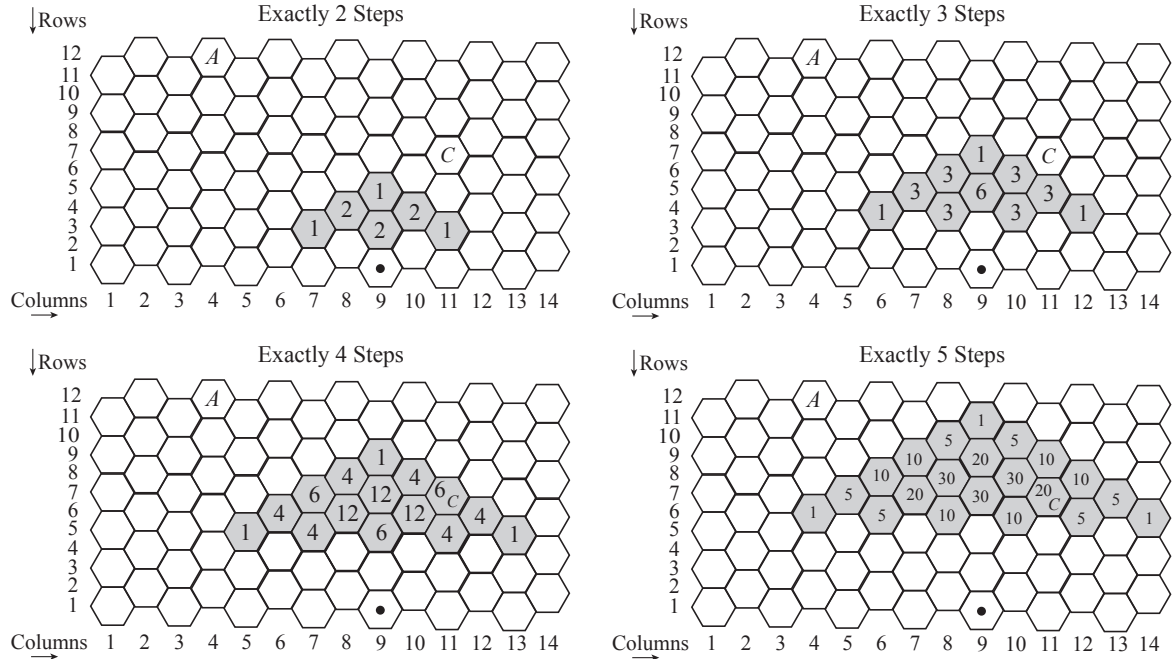
From the final diagram, we see that there are exactly 6 hexagons that can be reached in each 5 steps in at least 20 ways. In other words, $n = 6$.

Solution 2

We note that, when a fixed set of steps from a given path are rearranged, the endpoint of the path does not change.

This is because each step changes the row number and column number in a pre-determined way and so the outcome of each step is not influenced by the other steps in a path.

In other words, the paths ↗↖↑↑↗ and ↑↗↗↖↑ have the same endpoint.

Since there are 3 different kinds of steps (↖, ↑, ↗), a path with exactly 5 steps can be made up of

  (i) 5 of the same step, or

 (ii) 4 of one step and 1 of another, or

(iii) 3 of one step and 2 of another, or

 (iv) 3 of one step, 1 of a second type, and 1 of the third type, or

  (v) 2 of one step, 2 of a second type, and 1 of the third type.

These are the only possible combinations of different types of steps.

Each specific collection of 5 steps can possibly be re-arranged in a number of different ways.

We count the number of paths in each category:

  (i) 5 of the same step

       There are 3 choices of the type of step.

       For each choice, there is only 1 way to arrange the 5 steps.

Therefore, starting from r1c9, there is thus 1 path to each of r6c4 (5 $\nwarrow$ steps), r11c9 (5 $\uparrow$ steps), and r6c14 (5 $\nearrow$ steps).

(ii) 4 of one step and 1 of another

There are 3 choices for the type of step to be taken 4 times (we call this choice x) and, for each of these choices, there are 2 choices for the type of step to be taken 1 time (we call this choice y).

There are thus $3 \times 2 = 6$ choices for the endpoint of the path (each choice of x and y gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of x and y, there are 5 ways to arrange $xxxxy$ since there are 5 possible positions for the y, after which the x's fill the remaining spots.

Therefore, starting from r1c9, there are thus 5 paths to each of r7c5 ($x = \nwarrow$, $y = \uparrow$), r6c6 ($x = \nwarrow$, $y = \nearrow$), r7c13 ($x = \nearrow$, $y = \uparrow$), r6c12 ($x = \nearrow$, $y = \nwarrow$), r10c8 ($x = \uparrow$, $y = \nwarrow$), r10c10 ($x = \uparrow$, $y = \nearrow$).

(iii) 3 of one step and 2 of another

There are 3 choices for the type of step to be taken 3 times (we call this choice x) and, for each of these choices, there are 2 choices for the type of step to be taken 2 times (we call this choice y).

There are thus $3 \times 2 = 6$ choices for the endpoint of the path (each choice of x and y gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of x and y, there are 10 ways to arrange $xxxyy$ since there are 10 ways to choose the positions for the two y's:

  1st and 2nd, 1st and 3rd, 1st and 4th, 1st and 5th,
  2nd and 3rd, 2nd and 4th, 2nd and 5th,
  3rd and 4th, 3rd and 5th,
  4th and 5th

after which the x's fill the remaining spots.

Therefore, starting from r1c9, there are thus 10 paths to each of r8c6 ($x = \nwarrow$, $y = \uparrow$), r6c8 ($x = \nwarrow$, $y = \nearrow$), r8c12 ($x = \nearrow$, $y = \uparrow$), r6c10 ($x = \nearrow$, $y = \nwarrow$), r9c7 ($x = \uparrow$, $y = \nwarrow$), r9c11 ($x = \uparrow$, $y = \nearrow$).

(iv) 3 of one step, 1 of a second type, and 1 of the third type

There are 3 choices for the type of step to be taken 3 times (we call this choice x). After x is chosen, the types of steps to be taken 1 time each are fixed (we call these y and z in some order).

There are thus 3 choices for the endpoint of the path (each choice of x gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of x, y and z, there are 20 ways to arrange $xxxyz$ since there are 5 ways to choose the position for the y, after which there are 4 ways to choose the position for the z, after which the x's fill the remaining spots.

Therefore, starting from r1c9, there are thus 20 paths to each of r7c7 ($x = \nwarrow$, $y = \uparrow$, $z = \nearrow$), r7c11 ($x = \nearrow$, $y = \nearrow$, $z = \uparrow$), r9c9 ($x = \uparrow$, $y = \nwarrow$, $z = \nearrow$).

(v) 2 of one step, 2 of a second type, and 1 of the third type

There are 3 choices for the type of step to be taken 1 time (we call this choice z). After z is chosen, the types of steps to be taken 2 times each are fixed (we call these x and y).

There are thus 3 choices for the endpoint of the path (each choice of z gives a different endpoint and each re-arrangement of a specific set of steps gives the same endpoint).

Given a choice of x, y and z, there are 30 ways to arrange $xxyyz$ since there are 10 ways to choose the positions for the x's (as there were 10 ways to choose 2 of 5 positions for the y's above), after which there are 3 ways to choose the position for the z (there are three positions left), after which the y's fill the remaining spots. Therefore, starting from r1c9, there are thus 30 paths to each of r8c8 ($x =\nwarrow$, $y =\uparrow$, $z =\nearrow$), r7c9 ($x =\nearrow$, $y =\nwarrow$, $z =\uparrow$), r8c10 ($x =\uparrow$, $y =\nearrow$, $z =\searrow$).

Therefore, in summary, there are 6 distinct hexagons (r7c7, r7c11, r9c9, r8c8, r7c9, r8c10) which are the endpoints of at least 20 paths of 5 steps; in other words, $n = 6$.