The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2015 Cayley Contest

(Grade 10)

Tuesday, February 24, 2015
(in North America and South America)

Wednesday, February 25, 2015
(outside of North America and South America)

Solutions

1. Evaluating, $2 \times 2015 - 2015 = 4030 - 2015 = 2015$.
   Alternatively, $2 \times 2015 - 2015 = 2 \times 2015 - 1 \times 2015 = 1 \times 2015 = 2015$.

   ANSWER: (A)

2. Evaluating, $\sqrt{1} + \sqrt{9} = 1 + 3 = 4$.

   ANSWER: (D)

3. The volume of a rectangular box equals the area of its base times its height.
   Thus, the height equals the volume divided by the area of the base.
   The area of the base of the given box is $2 \cdot 5 = 10 \text{ cm}^2$.
   Therefore, the height of the given box is $\dfrac{30}{10} = 3$ cm.

   ANSWER: (C)

4. *Solution 1*
   $\angle SRQ$ is an exterior angle of $\triangle PQR$.
   Thus, $\angle SRQ = \angle RPQ + \angle PQR = 50^\circ + 90^\circ = 140^\circ$.
   Therefore, $x^\circ = 140^\circ$ and so $x = 140$.

   Solution 2
   The sum of the angles of $\triangle PQR$ is 180°, and so

   $$\angle PRQ = 180^\circ - \angle RPQ - \angle PQR = 180^\circ - 50^\circ - 90^\circ = 40^\circ$$

   Since $\angle PRQ$ and $\angle SRQ$ are supplementary, then $x^\circ + 40^\circ = 180^\circ$, and so $x = 180 - 40 = 140$.

   ANSWER: (D)

5. From the given graph, 3 provinces and territories joined Confederation between 1890 and 1929, and 1 between 1930 and 1969.
   Thus, between 1890 and 1969, a total of 4 provinces and territories joined Confederation.
   Therefore, if one of the 13 provinces and territories is chosen at random, the probability that it joined Confederation between 1890 and 1969 is $\frac{4}{13}$.

   ANSWER: (B)

6. Since $a^2 = 9$, then $a^4 = (a^2)^2 = 9^2 = 81$.
   Alternatively, we note that since $a^2 = 9$, then $a = \pm 3$. If $a = 3$, then $a^4 = 3^4 = 81$ and if $a = -3$, then $a^4 = (-3)^4 = 3^4 = 81$.

   ANSWER: (B)

7. First, we note that $3 + \frac{1}{10} + \frac{4}{100} = 3 + \frac{10}{100} + \frac{4}{100} = 3 + \frac{14}{100} = 3\frac{14}{100}$.
   Since $\frac{14}{100} = 0.14$, then the given expression also equals 3.14.
   Since $\frac{14}{100} = \frac{7}{50}$, then the given expression also equals $3\frac{7}{50}$.
   We also see that $3\frac{7}{50} = 3 + \frac{7}{50} = \frac{150}{50} + \frac{7}{50} = \frac{157}{50}$.
   Therefore, the only remaining expression is $3\frac{5}{110}$.
   We note further that $3\frac{5}{110} = 3.0\overline{45}$, which is not equal to 3.14.

   ANSWER: (C)

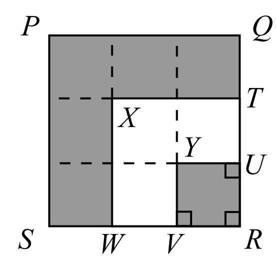
8. Violet starts with one-half of the money that she needed to buy the necklace.
After her sister gives her money, she has three-quarters of the amount that she needs.
This means that her sister gave her $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$ of the total amount that she needs.
Since she now has three-quarters of the amount that she needs, then she still needs one-quarter of the total cost.
In other words, her father will give her the same amount that her sister gave her, or \$30.

ANSWER: (D)

9. Since January 5 is a Monday and Mondays are 7 days apart, then January 12, 19 and 26 are also Mondays.
Since John goes for a run every 3 days, the dates in January on which he runs are January 5, 8, 11, 14, 17, 20, 23, 26, and 29.
The first of the Mondays on which John goes for a run after January 5 is January 26.

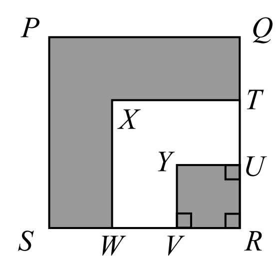
ANSWER: (C)

10. *Solution 1*



Since $PQRS$ is a square and TX and UY are perpendicular to QR, then TX and UY are parallel to PQ and SR.
Similarly, VY and WX are parallel to PS and QR.
Therefore, if we extend WX and VY to meet PQ and extend TX and UY to meet PS, then square $PQRS$ is divided into 9 rectangles.
Since $QT = TU = UR = 1$ and $RV = VW = WS = 1$, then in fact $PQRS$ is divided into 9 squares, each of which is 1 by 1.
Of these 9 squares, 6 are shaded and 3 are unshaded.
Therefore, the ratio of the shaded area to the unshaded area is $6 : 3$, which equals $2 : 1$.

Solution 2



Consider quadrilateral $YURV$.
$YURV$ has three right angles: at U and V because UY and VY are perpendicular to QR and RS, respectively, and at R because $PQRS$ is a square. Since $YURV$ has three right angles, then it has four right angles and so is a rectangle.
Since $RV = UR = 1$, then $YURV$ is actually a square and has side length 1, and so has area 1^2, or 1.
Similarly, $XTRW$ is a square of side length 2, and so has area 2^2, or 4.
Since square $PQRS$ is 3×3, then its area is 3^2, or 9.
The area of the unshaded region is equal to the difference between the areas of square $XTRW$ and square $YURV$, or $4 - 1 = 3$.
Since square $PQRS$ has area 9 and the area of the unshaded region is 3, then the area of the shaded region is $9 - 3 = 6$.
Finally, the ratio of the shaded area to the unshaded area is $6 : 3$, which equals $2 : 1$.

ANSWER: (A)

11. From the given definition,

$$4 \otimes 8 = \frac{4}{8} + \frac{8}{4} = \frac{1}{2} + 2 = 2\frac{1}{2} = \frac{5}{2}$$

ANSWER: (E)

12. The line with equation $y = \frac{3}{2}x + 1$ has slope $\frac{3}{2}$.
Since the line segment joining $(-1, q)$ and $(-3, r)$ is parallel to the line with equation $y = \frac{3}{2}x+1$, then the slope of this line segment is $\frac{3}{2}$.
Therefore, $\dfrac{r-q}{(-3)-(-1)} = \dfrac{3}{2}$ or $\dfrac{r-q}{-2} = \dfrac{3}{2}$.
Thus, $r - q = (-2) \cdot \dfrac{3}{2} = -3$.

Answer: (E)

13. *Solution 1*
The two teams include a total of $25 + 19 = 44$ players.
There are exactly 36 students who are at least one team.
Thus, there are $44 - 36 = 8$ students who are counted twice.
Therefore, there are 8 students who play both baseball and hockey.

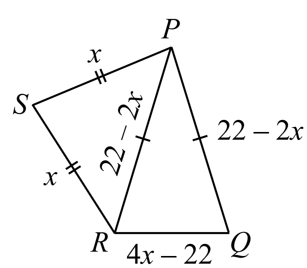
Solution 2
Suppose that there are x students who play both baseball and hockey.
Since there are 25 students who play baseball, then $25-x$ of these play baseball and not hockey.
Since there are 19 students who play hockey, then $19-x$ of these play hockey and not baseball.
Since 36 students play either baseball or hockey or both, then

$$(25 - x) + (19 - x) + x = 36$$

(The left side is the sum of the numbers of those who play baseball and not hockey, those who play hockey and not baseball, and those who play both.)
Therefore, $44 - x = 36$ and so $x = 44 - 36 = 8$.
Thus, 8 students play both baseball and hockey.

Answer: (B)

14. Since $PS = SR = x$ and the perimeter of $\triangle PRS$ is 22, then $PR = 22 - PS - SR = 22 - 2x$.
Since $PQ = PR$ and $PR = 22 - 2x$, then $PQ = 22 - 2x$.
Since $\triangle PQR$ has perimeter 22, then $RQ = 22-PR-PQ = 22-(22-2x)-(22-2x) = 4x-22$.



Since the perimeter of $PQRS$ is 24, then

$$\begin{aligned} PQ + RQ + SR + PS &= 24 \\ (22 - 2x) + (4x - 22) + x + x &= 24 \\ 4x &= 24 \\ x &= 6 \end{aligned}$$

Therefore, $x = 6$.

Answer: (D)

15. We note that

$$1! = 1 \qquad 2! = (1)(2) = 2 \qquad 3! = (1)(2)(3) = 6$$

$$4! = (1)(2)(3)(4) = 24 \qquad 5! = (1)(2)(3)(4)(5) = 120$$

Thus, $1! + 2! + 3! + 4! + 5! = 1 + 2 + 6 + 24 + 120 = 153$.
Now for each positive integer $n \geq 5$, the ones digit of $n!$ is 0:

> One way to see this is to note that we obtain each successive factorial by multiplying the previous factorial by an integer. (For example, $6! = 6(5!)$.)
> Thus, if one factorial ends in a 0, then all subsequent factorials will also end in a 0.
> Since the ones digit of $5!$ is 0, then the ones digit of each $n!$ with $n > 5$ will also be 0.
> Alternatively, we note that for each positive integer n, the factorial $n!$ is the product of the positive integers from 1 to n. When $n \geq 5$, the product represented by $n!$ includes factors of both 2 and 5, and so has a factor of 10, thus has a ones digit of 0.

Therefore, the ones digit of each of $6!$, $7!$, $8!$, $9!$, and $10!$ is 0, and so the ones digit of $6! + 7! + 8! + 9! + 10!$ is 0.
Since the ones digit of $1! + 2! + 3! + 4! + 5!$ is 3 and the ones digit of $6! + 7! + 8! + 9! + 10!$ is 0, then the ones digit of $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10!$ is $3 + 0$ or 3.
(We can verify, using a calculator, that $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10! = 4\,037\,913$.)

Answer: (B)

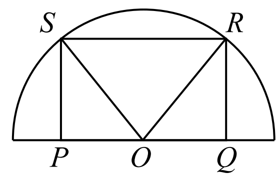
16. In a magic square, the numbers in each row, the numbers in each column, and numbers on each diagonal have the same sum.
Since the sum of the numbers in the first row equals the sum of the numbers in the first column, then $a + 13 + b = a + 19 + 12$ or $b = 19 + 12 - 13 = 18$.
Therefore, the sum of the numbers in any row, in any column, or along either diagonal equals the sum of the numbers in the third column, which is $18 + 11 + 16 = 45$.
Using the first column, $a + 19 + 12 = 45$ or $a = 14$.
Using the second row, $19 + c + 11 = 45$ or $c = 15$.
Thus, $a + b + c = 14 + 18 + 15 = 47$.

Answer: (C)

17. Suppose that Deanna drove at v km/h for the first 30 minutes.
Since 30 minutes equals one-half of an hour, then in these 30 minutes, she drove $\frac{1}{2}v$ km.
In the second 30 minutes, she drove at $(v + 20)$ km/h.
Thus, in these second 30 minutes, she drove $\frac{1}{2}(v + 20)$ km.
Since she drove 100 km in total, then $\frac{1}{2}v + \frac{1}{2}(v + 20) = 100$ or $\frac{1}{2}v + \frac{1}{2}v + 10 = 100$.
Thus, $v + 10 = 100$ or $v = 90$.
Therefore, Deanna drove 90 km/h for the first 30 minutes.

Answer: (B)

18. Let O be the centre of the circle. Join OS and OR.



Since the diameter of the semicircle is 20, then its radius is half of this, or 10.
Since OS and OR are radii, then $OS = OR = 10$.
Consider $\triangle OPS$ and $\triangle OQR$.
Since $PQRS$ is a rectangle, both triangles are right-angled (at P and Q).
Also, $PS = QR$ (equal sides of the rectangle) and $OS = OR$ (since they are radii of the circle).
Therefore, $\triangle OPS$ is congruent to $\triangle OQR$. (Right-angled triangles with equal hypotenuses and one other pair of equal corresponding sides are congruent.)
Since $\triangle OPS$ and $\triangle OQR$ are congruent, then $OP = OQ$.
Since $PQ = 16$, then $OP = \frac{1}{2}PQ = 8$.
Finally, since $\triangle OPS$ is right-angled at P, then we can apply the Pythagorean Theorem to conclude that $PS = \sqrt{OS^2 - OP^2} = \sqrt{10^2 - 8^2} = \sqrt{100 - 64} = 6$.

Answer: (A)

19. Consider a stack of bills with a total value of \$1000 that includes x \$20 bills and y \$50 bills.
The \$20 bills are worth \$$20x$ and the \$50 bills are worth \$$50y$, and so $20x + 50y = 1000$ or $2x + 5y = 100$.
Determining the number of possible stacks that the teller could have is equivalent to determining the numbers of pairs (x, y) of integers with $x \geq 1$ and $y \geq 1$ and $2x + 5y = 100$.
(We must have $x \geq 1$ and $y \geq 1$ because each stack includes at least one \$20 bill and at least one \$50 bill.)
Since $x \geq 1$, then $2x \geq 2$, so $5y = 100 - 2x \leq 98$.
This means that $y \leq \frac{98}{5} = 19.6$.
Since y is an integer, then $y \leq 19$.
Also, since $5y = 100 - 2x$, then the right side is the difference between two even integers, so $5y$ is itself even, which means that y must be even.
Therefore, the possible values of y are $2, 4, 6, 8, 10, 12, 14, 16, 18$.
Each of these values gives a pair (x, y) that satisfies the equation $2x + 5y = 100$:

$$(x, y) = (45, 2), (40, 4), (35, 6), (30, 8), (25, 10), (20, 12), (15, 14), (10, 16), (5, 18)$$

Translating back to the original context, we see that the maximum number of stacks that the teller could have is 9.

Answer: (A)

20. First, we calculate the value of $72\left(\frac{3}{2}\right)^n$ for each integer from $n=-3$ to $n=4$, inclusive:

$$72\left(\tfrac{3}{2}\right)^{-3} = 72\cdot\tfrac{2^3}{3^3} = 72\cdot\tfrac{8}{27} = \tfrac{64}{3} \qquad 72\left(\tfrac{3}{2}\right)^{-2} = 72\cdot\tfrac{2^2}{3^2} = 72\cdot\tfrac{4}{9} = 32$$
$$72\left(\tfrac{3}{2}\right)^{-1} = 72\cdot\tfrac{2^1}{3^1} = 72\cdot\tfrac{2}{3} = 48 \qquad 72\left(\tfrac{3}{2}\right)^{0} = 72\cdot 1 = 72$$
$$72\left(\tfrac{3}{2}\right)^{1} = 72\cdot\tfrac{3^1}{2^1} = 72\cdot\tfrac{3}{2} = 108 \qquad 72\left(\tfrac{3}{2}\right)^{2} = 72\cdot\tfrac{3^2}{2^2} = 72\cdot\tfrac{9}{4} = 162$$
$$72\left(\tfrac{3}{2}\right)^{3} = 72\cdot\tfrac{3^3}{2^3} = 72\cdot\tfrac{27}{8} = 243 \qquad 72\left(\tfrac{3}{2}\right)^{4} = 72\cdot\tfrac{3^4}{2^4} = 72\cdot\tfrac{81}{16} = \tfrac{729}{2}$$

Therefore, there are at least 6 integer values of n for which $72\left(\frac{3}{2}\right)^n$ is an integer, namely $n=-2,-1,0,1,2,3$.
Since 6 is the largest possible choice given, then it must be the correct answer (that is, it must be the case that there are no more values of n that work).

We can justify this statement informally by noting that if we start with $72\left(\frac{3}{2}\right)^4 = \frac{729}{2}$, then making n larger has the effect of continuing to multiply by $\frac{3}{2}$ which keeps the numerator odd and the denominator even, and so $72\left(\frac{3}{2}\right)^n$ is never an integer when $n>3$. A similar argument holds when $n<-2$.
We could justify the statement more formally by re-writing

$$72\left(\tfrac{3}{2}\right)^n = 3^2\cdot 2^3\cdot 3^n\cdot 2^{-n} = 3^2 3^n 2^3 2^{-n} = 3^{2+n}2^{3-n}$$

For this product to be an integer, it must be the case that each of 3^{2+n} and 2^{3-n} is an integer. (Each of 3^{2+n} and 2^{3-n} is either an integer or a fraction with numerator 1 and denominator equal to a power of 2 or 3. If each is such a fraction, then their product is less than 1 and so is not an integer. If exactly one is an integer, then their product equals a power of 2 divided by a power of 3 or vice versa. Such a fraction cannot be an integer since powers of 3 cannot be "divided out" of powers of 2 and vice versa.)
This means that $2+n\geq 0$ (and so $n\geq -2$) and $3-n\geq 0$ (and so $n\leq 3$).
Therefore, $-2\leq n\leq 3$. The integers in this range are the six integers listed above.

Answer: (E)

21. We are given that three consecutive odd integers have an average of 7.
These three integers must be 5, 7 and 9.
One way to see this is to let the three integers be $a-2, a, a+2$. (Consecutive odd integers differ by 2.)
Since the average of these three integers is 7, then their sum is $3\cdot 7 = 21$.
Thus, $(a-2)+a+(a+2)=21$ or $3a=21$ and so $a=7$.
When m is included, the average of the four integers equals their sum divided by 4, or $\dfrac{21+m}{4}$.
This average is an integer whenever $21+m$ is divisible by 4.
Since 21 is 1 more than a multiple of 4, then m must be 1 less than a multiple of 4 for the sum $21+m$ to be a multiple of 4.
The smallest positive integers m that are 1 less than a multiple of 4 are 3, 7, 11, 15, 19.
Since m cannot be equal to any of the original three integers 5, 7 and 9, then the three smallest possible values of m are 3, 11 and 15.
The sum of these possible values is $3+11+15=29$.

Answer: (D)

22. We label the players P, Q, R, S, T, U.
Each player plays 2 games against each of the other 5 players, and so each player plays 10 games. Thus, each player earns between 0 and 10 points, inclusive.
We show that a player must have at least $9\frac{1}{2}$ points to guarantee that he has more points than every other player.
We do this by showing that it is possible to have two players with 9 points, and that if one player has $9\frac{1}{2}$ or 10 points, then every other player has at most 9 points.
Suppose that P and Q each win both of their games against each of R, S, T, and U and tie each of their games against each other.
Then P and Q each have a record of 8 wins, 2 ties, 0 losses, giving them each $8 \cdot 1 + 2 \cdot \frac{1}{2} + 0 \cdot 0$ or 9 points.
We note also that R, S, T, U each have 4 losses (2 against each of P and Q), so have at most 6 points.
Therefore, if a player has 9 points, it does not guarantee that he has more points than every other player, since in the scenario above both P and Q have 9 points.
Suppose next that P has $9\frac{1}{2}$ or 10 points.
If P has 10 points, then P won every game that he played, which means that every other player lost at least 2 games, and so can have at most 8 points.
If P has $9\frac{1}{2}$ points, then P must have 9 wins, 1 tie and 0 losses. (With $9\frac{1}{2}$ points, P has only "lost" $\frac{1}{2}$ point and so cannot have lost any games and can only have tied 1 game.)
Since P has 9 wins, then P must have beaten each of the other players at least once. (If there was a player that P had not beaten, then P would have at most $4 \cdot 2 = 8$ wins.)
Since every other player has at least 1 loss, then every other player has at most 9 points.
Therefore, if P has $9\frac{1}{2}$ or 10 points, then P has more points than every other player.
In summary, if a player has $9\frac{1}{2}$ or 10 points, then he is guaranteed to have more points than every other player, while if he has 9 points, it is possible to have the same number of points as another player.
Thus, the minimum number of points necessary to guarantee having more points than every other player is $9\frac{1}{2}$.

Answer: (D)

23. Nylah's lights come on randomly at one of the times 7:00 p.m., 7:30 p.m., 8:00 p.m., 8:30 p.m., or 9:00 p.m., each with probability $\frac{1}{5}$.
What is the probability that the lights come on at 7:00 p.m. and are on for t hours with $4 < t < 5$?
If the lights come on at 7:00 p.m. and are on for between 4 and 5 hours, then they go off between 11:00 p.m. and 12:00 a.m.
Since the length of this interval is 1 hour and the length of the total interval of time in which the lights randomly go off is 2 hours (11:00 p.m. to 1:00 a.m.), then the probability that they go off between 11:00 p.m. and 12:00 a.m. is $\frac{1}{2}$.
Therefore, the probability that the lights come on at 7:00 p.m. and are on for t hours with $4 < t < 5$ is $\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$.
Similarly, if the lights come on at 7:30 p.m., they can go off between 11:30 p.m. and 12:30 a.m., and the probability of this combination of events is also $\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$.
Similarly again, the probability of the lights coming on at 8:00 p.m. and going off between 12:00 a.m. and 1:00 a.m. is also $\frac{1}{10}$.
If the lights come on at 8:30 p.m., then to be on for between 4 and 5 hours, they must go off between 12:30 a.m. and 1:00 a.m. (They cannot stay on past 1:00 a.m.)
The probability of this combination is $\frac{1}{5} \cdot \frac{1/2}{2} = \frac{1}{5} \cdot \frac{1}{4} = \frac{1}{20}$.

If the lights come on at 9:00 p.m., they cannot be on for more than 4 hours, since the latest that they can go off is 1:00 a.m.
Therefore, the probability that the lights are on for between 4 and 5 hours is $3 \cdot \frac{1}{10} + \frac{1}{20} = \frac{7}{20}$.
(We note that we can safely ignore the question of whether the lights coming on at 7:30 p.m. and going off at exactly 11:30 p.m., for example, affects the probability calculation, because 11:30 p.m. is a single point in an interval containing an infinite number of points, and so does not affect the probability.)

Answer: (E)

24. We call an arrangement of tiles of a specific region that satisfies any given conditions a *tiling*.
Since no tile can cross the line TU, we can consider tilings of the regions $PTUS$ and $TQRU$ separately.
We proceed without including the units of metres on each dimension.
First, we determine the number of tilings of the 2×4 region $PTUS$.
To be able to discuss this effectively, we split $PTUS$ into 8 squares measuring 1×1 and label these squares as shown in Figure 1.

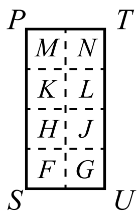


Figure 1

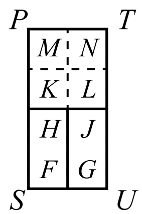


Figure 2

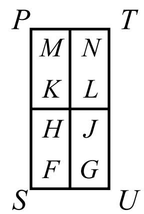


Figure 3

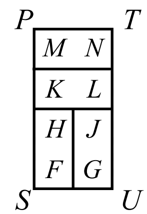


Figure 4

Consider square F. It must be covered with either a horizontal tile (covering FG) or a vertical tile (covering FH).
If F is covered with a vertical tile FH, then G must also be covered with a vertical tile GJ, since G is covered and its tile cannot overlap TU.
This gives the configuration in Figure 2.
The remaining 2×2 square can be covered in the two ways shown in Figure 3 and Figure 4.
This gives 2 tilings of $PTUS$ so far.
If F is covered with a horizontal tile FG, then we focus on H and J.
Either H and J are covered by one horizontal tile HJ (again leaving a 2×2 square that can be covered in 2 ways as above (see Figures 5 and 6)) or H and J are each covered by vertical tiles HK and JL, which means that MN is covered with 1 horizontal tile (see Figure 7).

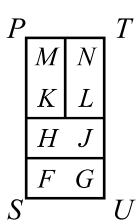


Figure 5

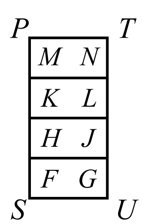


Figure 6

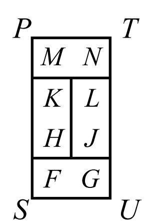


Figure 7

So if F is covered with a horizontal tile, there are $2 + 1 = 3$ tilings.
In total, there are $2 + 3 = 5$ possible tilings of the 2×4 region $PTUS$.

Consider now region $TQRU$.
Suppose that the number of tilings of the 4×4 region $TQRU$ is t.
Then for each of the 5 tilings of $PTUS$, there are t tilings of $TQRU$, so there will be $5t$ tilings of the entire region $PQRS$.

Divide $TQRU$ into 1×1 squares and label them as shown in Figure 8. We define V and W to be the midpoints of TQ and UR, respectively.

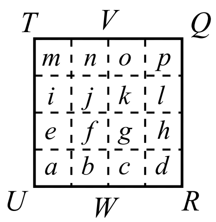


Figure 8

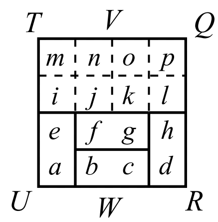


Figure 9

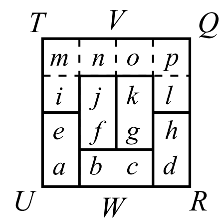


Figure 10

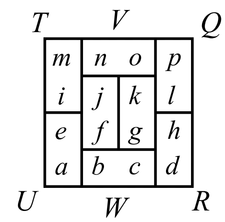


Figure 11

We consider two cases – either the line VW is overlapped by a tile, or it isn't.
If VW is not overlapped by a tile, then each of $TVWU$ and $VQRW$ is a 2×4 region to be tiled, and so can be tiled in 5 ways, as we saw with $PTUS$.
In this case, the number of tilings of $TQRU$ is $5 \times 5 = 25$.

Suppose that VW is overlapped by at least one tile.
If bc is covered by a horizontal tile, then ae and dh are covered by vertical tiles.
In this case, either fg is covered by a horizontal tile (Figure 9), or each of f and g is covered by a vertical tile (Figure 10).
In the first case (Figure 9), the upper 4×2 region needs to be tiled and there are 5 ways to do this, as above.
In the second case (Figure 10), the remaining tiling is forced to be as shown in Figure 11. Can you see why?
Therefore, if bc is covered by a horizontal tile, there are $5 + 1 = 6$ tilings.

Suppose that bc is not covered by a horizontal tile, but fg is covered by a horizontal tile.
Then ab and cd are each covered by horizontal tiles and so ei and hl are each covered by vertical tiles and so mn and op are each covered by horizontal tiles, and so jk must be covered by a horizontal tile.

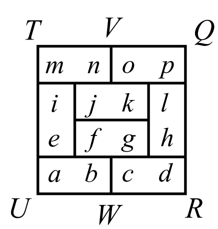


Figure 12

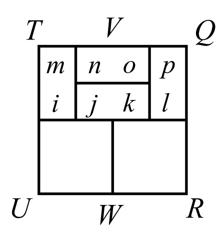


Figure 13

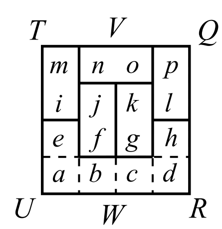


Figure 14

In other words, there is only 1 tiling in this case, shown in Figure 12.

Suppose now that bc and fg are not covered by a horizontal tile, but jk is.
In this case, each of the bottom 2×2 squares is tiled in a self-contained way. There are 2 ways to tile each, and so 4 ways to tile the pair of squares. (These tilings are each self-contained because if either ei or hl is covered by a vertical tile, then the remaining three tiles in the corresponding 2×2 square cannot be covered with 1×2 tiles.)

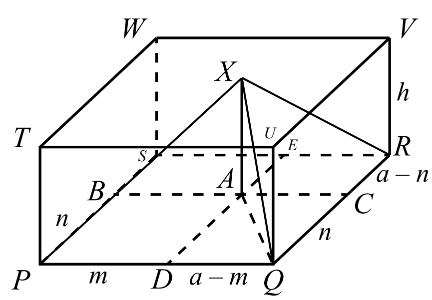
Furthermore, im and lp must be covered by vertical tiles, meaning that no is tiled with a horizontal tile, as shown in Figure 13, and so there is only one tiling of the upper rectangle.
Thus, there are $2 \times 2 \times 1 = 4$ tilings of $TQRU$ in this case, since the rest of the tiling is determined without choice.

Suppose finally that none of bc, fg, or jk is covered by a horizontal tile, but no is.
Then im and lp are covered with vertical tiles, which means that fj and gk are covered by vertical tiles, giving the diagram in Figure 14.
There is no way to complete this tiling without using a horizontal tile bc.
Therefore, there are no tilings in this case.

Finally, we can now say that there are $t = 25 + 6 + 4 + 1$ tilings of $TQRU$.
This means that there are $5 \times 36 = 180$ ways of tiling the entire 6×4 region with the given conditions.

Answer: (A)

25. Suppose that the square base $PQRS$ of the prism has side length a, and that the prism has height h.
We are asked to find the maximum possible area for rectangle $PQUT$.
The area of rectangle $PQUT$ is equal to ah, since $PQ = a$ and $QU = h$.
Let A be the point on $PQRS$ directly below X (that is, XA is perpendicular to the plane of $PQRS$). Note that $AX = h$.
Draw line segment BC through A with B on PS and C on QR so that BC is parallel to PQ.
Draw line segment DE through A with D on PQ and E on SR so that DE is parallel to QR.
Then segments BC and DE divide square $PQRS$ into four rectangles.
Let $PD = m$ and $PB = n$.
Then $SE = m$, $QC = n$, $DQ = ER = a - m$, and $CR = BS = a - n$.



$\triangle XAQ$ is right-angled at A so $QX^2 = AX^2 + AQ^2$.
But AQ is the hypotenuse of right-angled $\triangle ADQ$, so $AQ^2 = AD^2 + DQ^2$.
Thus, $QX^2 = AX^2 + AD^2 + DQ^2$.
Since $QX = 10$, $AX = h$, $AD = CQ = n$, and $DQ = a - m$, then $10^2 = h^2 + n^2 + (a - m)^2$.
Similarly, using $PX = 12$, we find that $12^2 = h^2 + n^2 + m^2$ and using $RX = 8$, we find that $8^2 = h^2 + (a - n)^2 + (a - m)^2$.
Subtracting $10^2 = h^2+n^2+(a-m)^2$ from $12^2 = h^2+n^2+m^2$, we obtain $144-100 = m^2-(a-m)^2$ or $44 = m^2 - (a^2 - 2am + m^2)$ which gives $44 = 2am - a^2$ or $m = \dfrac{44 + a^2}{2a}$.
Similarly, subtracting $8^2 = h^2 + (a - n)^2 + (a - m)^2$ from $10^2 = h^2 + n^2 + (a - m)^2$ gives $100 - 64 = n^2 - (a - n)^2$ or $36 = n^2 - (a^2 - 2an + n^2)$ which gives $36 = 2an - a^2$ or $n = \dfrac{36 + a^2}{2a}$.

Substituting these expressions for m and n into $12^2 = h^2 + n^2 + m^2$ gives

$$\begin{aligned} h^2 &= 144 - m^2 - n^2 \\ h^2 &= 144 - \left(\frac{44+a^2}{2a}\right)^2 - \left(\frac{36+a^2}{2a}\right)^2 \end{aligned}$$

Recall that we want to maximize ah.
Since $ah > 0$, then maximizing ah is equivalent to maximizing $(ah)^2 = a^2h^2$, which is equivalent to maximizing $4a^2h^2$.
From above,

$$\begin{aligned} 4a^2h^2 &= 4a^2\left(144 - \left(\frac{44+a^2}{2a}\right)^2 - \left(\frac{36+a^2}{2a}\right)^2\right) \\ &= 4a^2\left(144 - \frac{(44+a^2)^2}{4a^2} - \frac{(36+a^2)^2}{4a^2}\right) \\ &= 576a^2 - (44+a^2)^2 - (36+a^2)^2 \\ &= 576a^2 - (1936 + 88a^2 + a^4) - (1296 + 72a^2 + a^4) \\ &= -2a^4 + 416a^2 - 3232 \\ &= -2(a^4 - 208a^2 + 1616) \\ &= -2((a^2-104)^2 + 1616 - 104^2) \qquad \text{(completing the square)} \\ &= -2(a^2-104)^2 - 2(1616 - 104^2) \\ &= -2(a^2-104)^2 + 18400 \end{aligned}$$

Since $(a^2 - 104)^2 \geq 0$, then $4a^2h^2 \leq 18400$ (with equality when $a = \sqrt{104}$).
Therefore, $a^2h^2 \leq 4600$ and so $ah \leq \sqrt{4600}$.
This means that maximum possible area of $PQUT$ is $\sqrt{4600} = 10\sqrt{46} \approx 67.823$.
Of the given answers, this is closest to 67.82.

ANSWER: (B)