The CENTRE for EDUCATION
in MATHEMATICS and COMPUTING
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2015 Canadian Intermediate Mathematics Contest

Wednesday, November 25, 2015
(in North America and South America)

Thursday, November 26, 2015
(outside of North America and South America)

Solutions

Part A

1. *Solution 1*

Since $\frac{1000}{12} \approx 83.33$, then the largest multiple of 12 less than 1000 is $83 \times 12 = 996$.
Therefore, if Stephanie filled 83 cartons, she would have broken 4 eggs.
If Stephanie filled fewer than 83 cartons, she would have broken more than 12 eggs. If Stephanie filled more than 83 cartons, she would have needed more than 1000 eggs.
Thus, $n = 4$.

Solution 2

Since $80 \times 12 = 960$, then filling 80 cartons would leave $1000 - 960 = 40$ eggs left over.
Since $81 \times 12 = 972$, then filling 81 cartons would leave $1000 - 972 = 28$ eggs left over.
Since $82 \times 12 = 984$, then filling 82 cartons would leave $1000 - 984 = 16$ eggs left over.
Since $83 \times 12 = 996$, then filling 83 cartons would leave $1000 - 996 = 4$ eggs left over.
Since $84 \times 12 = 1008$, then Stephanie could not have filled 84 cartons.
Therefore, Stephanie must have filled 83 cartons and broken 4 eggs, and so $n = 4$.

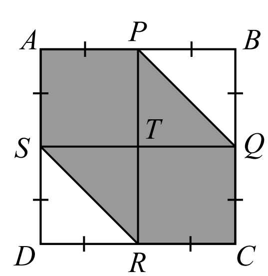
ANSWER: $n = 4$

2. *Solution 1*

Since the side length of the square is 4, then $AB = BC = CD = DA = 4$.
Therefore, the area of the square is $4 \times 4 = 16$.
Since P, Q, R, and S are the midpoints of sides, then each of AP, PB, BQ, QC, CR, RD, DS, and SA has length 2.
Now, the shaded area equals the area of the entire square minus the areas of $\triangle PBQ$ and $\triangle RDS$.
Since $ABCD$ is a square, then $\angle PBQ = 90^\circ$.
Thus, the area of $\triangle PBQ$ equals $\frac{1}{2}(PB)(BQ) = \frac{1}{2}(2)(2) = 2$.
Similarly, the area of $\triangle RDS$ is 2.
Finally, the shaded area equals $16 - 2 - 2 = 12$.

Solution 2

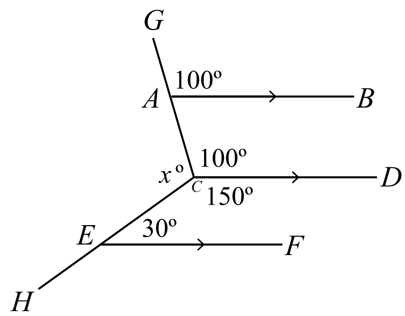
Join P to R and S to Q. Let the point of intersection of PR and SQ be T.
Since $ABCD$ is a square with side length 4, then joining the midpoints of opposite sides partitions $ABCD$ into four identical squares, each of side length 2. Each of these four squares has area $2^2 = 4$.



Therefore, the shaded area consists of square $APTS$, square $TQCR$, half of square $PBQT$, and half of square $STRD$. (Each of these last two squares has half of its area shaded because its diagonal divides it into two identical triangles.)
Therefore, the shaded area equals $4 + 4 + \frac{1}{2}(4) + \frac{1}{2}(4) = 12$.

ANSWER: 12

3. Since CD is parallel to EF, then $\angle FEC + \angle ECD = 180^\circ$. (These angles are co-interior angles.) Thus, $\angle ECD = 180^\circ - \angle FEC = 180^\circ - 30^\circ = 150^\circ$. Since AB is parallel to CD, then $\angle GCD = \angle GAB = 100^\circ$. (These angles are corresponding angles.)



Since $\angle ACE$, $\angle ECD$, and $\angle ACD$ completely surround C, then

$$\angle ACE + \angle ECD + \angle ACD = 360^\circ$$

Thus, $x^\circ + 150^\circ + 100^\circ = 360^\circ$ or $x + 250 = 360$ and so $x = 110$.

Answer: $x = 110$

4. Since $12x = 4y + 2$, then $4y - 12x = -2$ or $2y - 6x = -1$. Therefore, $6y - 18x + 7 = 3(2y - 6x) + 7 = 3(-1) + 7 = 4$.

Answer: 4

5. *Solution 1*
We start by finding the prime factorization of $6048(28^n)$.
Note that $28 = 4(7) = 2^2 7^1$.
Therefore, $28^n = (2^2 7^1)^n = 2^{2n} 7^n$.
Also, $6048 = 8(756) = 8(4)(189) = 8(4)(7)(27) = 2^3(2^2)(7^1)(3^3) = 2^5 3^3 7^1$.
Thus, $6048(28^n) = 2^5 3^3 7^1 (2^{2n} 7^n) = 2^{2n+5} 3^3 7^{n+1}$.
This is a perfect cube exactly when the exponent of each of the prime factors is a multiple of 3.
Since the exponent on the prime 3 is a multiple of 3, the problem is equivalent to asking for the largest positive integer n less than 500 for which each of $2n + 5$ and $n + 1$ is a multiple of 3.
If $n = 499$, these are 1003 and 500; neither is divisible by 3.
If $n = 498$, these are 1001 and 499; neither is divisible by 3.
If $n = 497$, these are 99 and 498; both are divisible by 3.
Thus, $n = 497$ is the largest positive integer less than 500 for which $6048(28^n)$ is a perfect cube.

Solution 2
We note that $6048 = 28(216)$.
Therefore, $6048(28^n) = 216(28)(28^n) = 216(28^{n+1}) = 6^3 28^{n+1}$, since $6^3 = 216$.
Since 6^3 is a perfect cube, then $6048(28^n)$ is a perfect cube exactly when 28^{n+1} is perfect cube.
Since 28 is itself not a perfect cube, then 28^{n+1} is a perfect cube exactly when $n+1$ is a multiple of 3.
Thus, we are looking for the largest positive integer $n < 500$ for which $n + 1$ is a multiple of 3.
Since 500 and 499 are not multiples of 3, but 498 is a multiple of 3 (since $498 = 3(166)$) we

have $n+1=498$ or $n=497$.
Therefore, $n=497$ is the largest positive integer less than 500 for which $6048(28^n)$ is a perfect cube.

ANSWER: 497

6. We want to determine the number of triples (a,b,c) of positive integers that satisfy the conditions $1 \leq a < b < c \leq 2015$ and $a+b+c=2018$.
Since $a<b<c$, then $a+b+c > a+a+a = 3a$.
Since $a+b+c=2018$ and $3a < a+b+c$, then $3a < 2018$ and so $a < \frac{2018}{3} = 672\frac{2}{3}$.
Since a is an integer, then $a \leq 672$. Therefore, $1 \leq a \leq 672$.
We proceed to consider the possible values of a systematically by considering first the cases when a is odd and then the cases when a is even.

When $a=1$, we want to find all pairs (b,c) of positive integers with $1 < b < c \leq 2015$ and $b+c = 2018-a = 2017$.
Since $1<b$, then $b \geq 2$.
Since $b<c$, then $b+c > b+b$ and so $2017 > 2b$, which gives $b < \frac{2017}{2} = 1008\frac{1}{2}$.
Since b is an integer, then $b \leq 1008$.
Therefore, b satisfies $2 \leq b \leq 1008$.
For each b, there is a corresponding c (namely, $c = 2017-b$).
Since there are 1007 values of b in this range, then there are 1007 triples in this case. These are $(1,2,2015)$, $(1,3,2014)$, ..., $(1,1008,1009)$.
When $a=3$, we want to find all pairs (b,c) of positive integers with $3 < b < c \leq 2015$ and $b+c = 2018-a = 2015$.
Since $3<b$, then $b \geq 4$.
Since $b<c$, then $b+c > b+b$ and so $2015 > 2b$, which gives $b < \frac{2015}{2} = 1007\frac{1}{2}$.
Since b is an integer, then $b \leq 1007$.
Therefore, b satisfies $4 \leq b \leq 1007$.
For each b, there is a corresponding c (namely, $c = 2015-b$).
Since there are 1004 values of b in this range, then there are 1004 triples in this case. These are $(3,4,2011)$, $(3,5,2010)$, ..., $(3,1007,1008)$.
In general, when $a=2k+1$ for some non-negative integer k, we want to find all pairs (b,c) of positive integers with $2k+1 < b < c \leq 2015$ and $b+c = 2018-a = 2017-2k$.
Since $2k+1 < b$, then $b \geq 2k+2$.
Since $b<c$, then $b+c > b+b$ and so $2017-2k > 2b$, which gives $b < \frac{2017-2k}{2} = (1008-k)+\frac{1}{2}$.
Since b is an integer, then $b \leq 1008-k$.
Therefore, b can take the values in the range $2k+2 \leq b \leq 1008-k$.
For each b, there is a corresponding c (namely, $c = 2017-2k-b$).
Since there are $(1008-k)-(2k+2)+1 = 1007-3k$ values of b in this range, then there are $1007-3k$ triples in this case.
Since $a \leq 672$, then the largest odd permissible value of a is $a=671$ which gives $2k+1=671$ or $k=335$.
Thus, k takes the integer values from $k=0$ (which gives $a=1$) to $k=335$ (which gives $a=671$), inclusive.
The number of triples in these cases range from 1007 (when $k=0$) to 2 (when $k=335$). The number of triples decreases by 3 when the value of k increases by 1.
Therefore, when a is odd, there are $1007+1004+1001+\cdots+8+5+2$ triples that satisfy the condition.

Now, we consider the cases when a is even.
When $a = 2$, we want to find all pairs (b, c) of positive integers with $2 < b < c \leq 2015$ and $b + c = 2018 - a = 2016$.
Since $2 < b$, then $b \geq 3$.
Since $b < c$, then $b + c > b + b$ and so $2016 > 2b$, which gives $b < 1008$.
Since b is an integer, then $b \leq 1007$.
Therefore, b can take the values in the range $3 \leq b \leq 1007$.
For each b, there is a corresponding c (namely, $c = 2016 - b$).
Since there are 1005 values of b in this range, then there are 1005 triples in this case. These are $(2, 3, 2013)$, $(2, 4, 2012)$, ..., $(2, 1007, 1009)$.
In general, when $a = 2m$ for some positive integer m, we want to find all pairs (b, c) of positive integers with $2m < b < c \leq 2015$ and $b + c = 2018 - a = 2018 - 2m$.
Since $2m < b$, then $b \geq 2m + 1$.
Since $b < c$, then $b + c > b + b$ and so $2018 - 2m > 2b$, which gives $b < 1009 - m$.
Since b is an integer, then $b \leq 1008 - m$.
Therefore, b can take the values in the range $2m + 1 \leq b \leq 1008 - m$.
For each b, there is a corresponding c (namely, $c = 2018 - 2m - b$).
Note that we need to have $2m + 1 \leq 1008 - m$ or $3m \leq 1007$ and so $m \leq 335\frac{2}{3}$; since m is an integer, we thus need $m \leq 335$.
Since there are $(1008 - m) - (2m + 1) + 1 = 1008 - 3m$ values of b in this range, then there are $1008 - 3m$ triples in this case.
Since $a \leq 672$, then the largest even permissible value of a is $a = 672$ which gives $2m = 672$ or $m = 336$. From above, we know further that $m \leq 335$.
Thus, m takes the integer values from $m = 1$ (which gives $a = 2$) to $m = 335$ (which gives $a = 670$), inclusive.
The number of triples in these cases range from 1005 (when $m = 1$) to 3 (when $m = 335$). The number of triples decreases by 3 when the value of m increases by 1.
Therefore, when a is even, there are $1005 + 1002 + 999 + \cdots + 6 + 3$ triples that satisfy the conditions.

Finally, the total number of triples that satisfy the given conditions is thus the sum of

$$1007 + 1004 + 1001 + \cdots + 8 + 5 + 2$$

and

$$1005 + 1002 + 999 + \cdots + 6 + 3 + 0$$

The first series is an arithmetic series with first term 1007, last term 2, and including 336 terms in total. (This is because the values of k are from 0 to 335, inclusive.)
One way to determine the sum of an arithmetic series is to multiply the number of terms by one-half and then multiply by the sum of the first and last terms.
Thus, the sum of this series is $\frac{336}{2}(1007 + 2)$.
The second series is an arithmetic series with first term 1005, last term 0, and including 336 terms in total. (This is because the values of m are from 1 to 336, inclusive.)
Thus, its sum is $\frac{335}{2}(1005 + 3)$.
Therefore, the total number of triples is

$$\frac{336}{2}(1007 + 2) + \frac{335}{2}(1005 + 3) = 168(1009) + 335(504) = 338\,352$$

ANSWER: 338 352

Part B

1. (a) *Solution 1*

Since 41 students are in the drama class and 15 students are in both drama and music, then $41 - 15 = 26$ students are in the drama class but not in the music class.
Since 28 students are in the music class and 15 students are in both drama and music, then $28 - 15 = 13$ students are in the music class but not in the drama class.
Therefore, there are $26 + 13 = 39$ students in exactly one class and 15 students in two classes.
Thus, there are $39 + 15 = 54$ students enrolled in the program.

Solution 2

If we add the number of students in the drama class and the number of students in the music class, we count the students in both classes twice.
Therefore, to obtain the total number of students enrolled in the program, we add the number of students in the drama class and the number of students in the music class, and subtract the number of students in both classes. (This has the effect of counting the students in the intersection only once.)
Thus, the number of students in the program is $41 + 28 - 15 = 54$.

(b) *Solution 1*

As in (a), there are $(3x - 5) - x = 2x - 5$ students enrolled in the drama class but not in the music class.
Similarly, there are $(6x + 13) - x = 5x + 13$ students enrolled in the music class but not in the drama class.
Therefore, there are $(2x - 5) + (5x + 13) = 7x + 8$ students enrolled in exactly one class and x students in two classes.
Since there were a total of 80 students enrolled in the program, then $(7x + 8) + x = 80$ or $8x = 72$ and so $x = 9$.

Solution 2

Using the method from Solution 2 to part (a), we obtain the equation $(3x - 5) + (6x + 13) - x = 80$.
Simplifying, we obtain $8x + 8 = 80$ or $8x = 72$, and so $x = 9$.

(c) Suppose that the number of students that were in both classes was a.
Since half of the students in the drama class were in both classes, there were also a students in the drama class who were not in the music class.
Since one-quarter of the students in the music class were in both classes, then three-quarters of the students in the music class were in music only. Thus, the number of students in music only was three times the number of students in both classes, or $3a$.
Therefore, there were $a + 3a = 4a$ students in exactly one class and a students in two classes.
Since a total of N students were in the program, then $4a + a = N$ or $N = 5a$.
This tells us that N is a multiple of 5, since a must be an integer.
Since N is between 91 and 99 and is a multiple of 5, then $N = 95$.

2. (a) The total length of the race is $2 + 40 + 10 = 52$ km.
When Emma has completed $\frac{1}{13}$ of the total distance, she has travelled $\frac{1}{13} \times 52 = 4$ km.

(b) Since Conrad completed the 2 km swim in 30 minutes (which is half an hour), then his speed was $2 \div \frac{1}{2} = 4$ km/h.
Since Conrad biked 12 times as fast as he swam, then he biked at $12 \times 4 = 48$ km/h.
Since Conrad biked 40 km, then the bike portion took him $\frac{40}{48} = \frac{5}{6}$ hours.
Since 1 hour equals 60 minutes, then the bike portion took him $\frac{5}{6} \times 60 = 50$ minutes.
Since Conrad ran 3 times as fast as he swam, then he ran at $3 \times 4 = 12$ km/h.
Since Conrad ran 10 km, then the running portion took him $\frac{10}{12} = \frac{5}{6}$ hours. This is again 50 minutes.
Therefore, the race took him $30 + 50 + 50 = 130$ minutes, or 2 hours and 10 minutes.
Since Conrad began the race at 8:00 a.m., then he completed the race at 10:10 a.m.

(c) Suppose that Alistair passed Salma after t minutes of the race.
Since Alistair swam for 36 minutes, then he had biked for $t - 36$ minutes (or $\dfrac{t-36}{60}$ hours) when they passed.
Since Salma swam for 30 minutes, then she had biked for $t - 30$ minutes (or $\dfrac{t-30}{60}$ hours) when they passed.
When Alistair and Salma passed, they had travelled the same total distance.
At this time, Alistair had swum 2 km. Since he bikes at 28 km/h, he had biked $28 \times \dfrac{t-36}{60}$ km.
Similarly, Salma had swum 2 km. Since she bikes at 24 km/h, she had biked $24 \times \dfrac{t-30}{60}$ km.
Since their total distances are the same, then

$$\begin{aligned} 2 + 28 \times \frac{t-36}{60} &= 2 + 24 \times \frac{t-30}{60} \\ 28 \times \frac{t-36}{60} &= 24 \times \frac{t-30}{60} \\ 28(t-36) &= 24(t-30) \\ 7(t-36) &= 6(t-30) \\ 7t - 252 &= 6t - 180 \\ t &= 72 \end{aligned}$$

Therefore, Alistair passed Salma 72 minutes into the race.
Since the race began at 8:00 a.m., then he passed her at 9:12 a.m.

Alternatively, we could notice that Salma started the bike portion of the race 6 minutes before Alistair.
Since 6 minutes is $\frac{1}{10}$ of an hour, she biked a distance of $\frac{1}{10} \times 24 = 2.4$ km before Alistair started riding.
In other words, she has a 2.4 km head start on Alistair.
Since Alistair bikes 4 km/h faster than Salma, it will take $\frac{2.4}{4} = 0.6$ hours for him to make up this difference.
Since 0.6 hours equals $0.6 \times 60 = 36$ minutes and he swam for 36 minutes before starting the bicycle portion, he will catch up a total of $36 + 36 = 72$ minutes into the race.

3. (a) *Solution 1*

Our strategy is to calculate the area of $\triangle ABC$ in two different ways, which will give us an equation for h.
Since $\triangle ABC$ has side lengths 20, 99 and 101, its semi-perimeter is $s = \frac{1}{2}(20 + 99 + 101)$ which equals $\frac{1}{2}(220)$ or 110.
By Heron's formula, the area of $\triangle ABC$ is thus

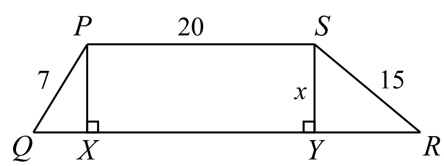
$$\begin{aligned}\sqrt{110(110-20)(110-99)(110-101)} &= \sqrt{110(90)(11)(9)} \\ &= \sqrt{11(10)(9)(10)(11)(9)} \\ &= \sqrt{9^2 10^2 11^2} \\ &= 9 \cdot 10 \cdot 11 \\ &= 990\end{aligned}$$

Also, $\triangle ABC$ can be viewed as having base 99 and height h.
Thus, the area of $\triangle ABC$ also equals $\frac{1}{2}(99)h$.
Equating the two representations of the area, we obtain $\frac{1}{2}(99)h = 990$ or $\frac{1}{2}h = 10$ from which $h = 20$.

Solution 2

Since $20^2 + 99^2 = 101^2$, then $\triangle ABC$ is right-angled at B.
Therefore, AB is the perpendicular height of the triangle from A to BC.
Thus, $h = 20$.

(b) As in (a), we calculate the area of the figure in two different ways.
Since $PQRS$ is a trapezoid with parallel sides $PS = 20$ and $QR = 40$ and the distance between the parallel sides is x, then the area of $PQRS$ is $\frac{1}{2}(20 + 40)x = 30x$.
To determine a second expression for the area, we drop perpendiculars from P and S to points X and Y on QR.

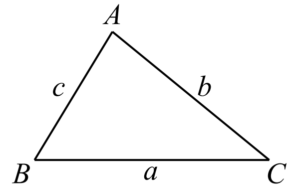


Then $PSYX$ is a rectangle since PS and XY are parallel and the angles at X and Y are right angles.
The area of rectangle $PSYX$ is $20x$.
If we cut out rectangle $PSYX$ and "glue" the two small triangles together along PX and SY (which are equal in length and each perpendicular to the base), we obtain a triangle with side lengths $PQ = 7$, $SR = 15$, and 20. (The third side has length 20 since we removed a segment of length 20 from a side of length 40.)
This triangle has semi-perimeter $\frac{1}{2}(7 + 15 + 20) = 21$ and so, using Heron's formula, its area is

$$\sqrt{21(21-7)(21-15)(21-20)} = \sqrt{21(14)(6)} = \sqrt{2^2 3^2 7^2} = 42$$

The area of trapezoid $PQRS$ is equal to the area of the rectangle plus the area of the triangle, or $20x + 42$.
Equating expressions for the area, we obtain $30x = 20x + 42$, which gives $10x = 42$ and so $x = 4.2$.

(c) Consider a triangle that has the given properties.
Call the side lengths a, b, c, for some integers a, b and c, with $a \le b \le c$.
Since the lengths of the two shortest sides differ by 1, then $b = a + 1$.
Thus, we call the side lengths a, $a + 1$, and c.
In a given triangle, each side must be less than half of the perimeter of the triangle. This is a result called the *Triangle Inequality*.



For example, in the triangle shown, the shortest path from B to C is the straight line segment of length a. Therefore, the path from B to A to C (of length $c + b$) must be longer than a.
This means that $c + b > a$ and so the perimeter $a + b + c$ is greater than $a + a$, or a is less than half of the perimeter. The same logic applies to each of the other sides too.
The semi-perimeter of the triangle with side lengths a, $a + 1$, c is $s = \frac{1}{2}(a + a + 1 + c)$.
Since the longest side length, c, and the semi-perimeter, s, differ by 1 and since c must be less than s, then $c = s - 1$ or $s = c + 1$.
Therefore, $c + 1 = \frac{1}{2}(a + a + 1 + c)$ and so $2c + 2 = 2a + 1 + c$ or $c = 2a - 1$.
Thus, the side lengths of the triangle are a, $a + 1$, and $2a - 1$.
The perimeter of the triangle, in terms of a, is thus $a + (a + 1) + (2a - 1) = 4a$, and so the semi-perimeter is $2a$.
Since the perimeter of the triangle is less than 200, then $4a < 200$ or $a < 50$.
We have now used four of the five conditions, and still need to use the fact that the area of the triangle is an integer.
Since the semi-perimeter is $2a$, by Heron's formula, the area of the triangle is

$$\begin{aligned}\sqrt{s(s-a)(s-b)(s-c)} &= \sqrt{2a(2a-a)(2a-(a+1))(2a-(2a-1))} \\ &= \sqrt{2a(a)(a-1)(1)} \\ &= \sqrt{a^2(2a-2)}\end{aligned}$$

For the area to be an integer, $\sqrt{a^2(2a-2)}$ must be an integer.
For $\sqrt{a^2(2a-2)}$ to be an integer, $a^2(2a-2)$ must be a perfect square.
For $a^2(2a-2)$ to be a perfect square, then $2a - 2$ must be a perfect square.
Now, $2a - 2$ is a multiple of 2 and so is even. Also, since a is a positive integer with $1 \le a < 50$, then $2a - 2$ is an integer with $0 \le 2a - 2 < 98$.
The even perfect squares in this range are 0, 4, 16, 36, 64.
In these cases, we have $a = 1, 3, 9, 19, 33$.
Since the triangle has side lengths a, $a + 1$, and $2a - 1$, then this gives triangles with side lengths

$$1, 2, 1 \quad 3, 4, 5 \quad 9, 10, 17 \quad 19, 20, 37 \quad 33, 34, 65$$

The first set of integers are not the side lengths of a triangle, since $1 + 1 = 2$.
Each of the four remaining sets of integers do form a triangle, since each side length is less than the sum of the other two side lengths.

Therefore, the four triangles that satisfy the five given conditions have side lengths

$$3, 4, 5 \quad 9, 10, 17 \quad 19, 20, 37 \quad 33, 34, 65$$