Canadian Mathematics Competition

An activity of the Centre for Education in Mathematics and Computing, University of Waterloo, Waterloo, Ontario

2005 Euclid Contest

Tuesday, April 19, 2005

Solutions

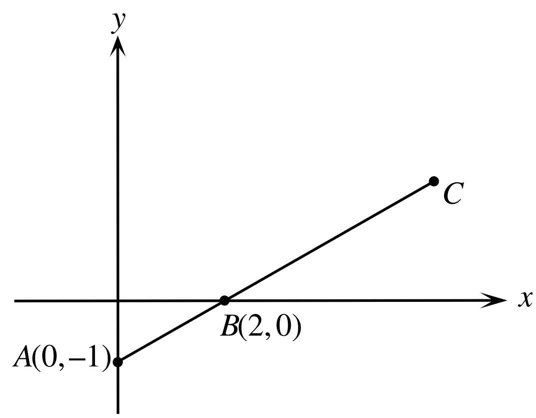
1. (a) Answer: $a = 5$

Since (a, a) lies on the line $3x - y = 10$, then $3a - a = 10$ or $2a = 10$ or $a = 5$.

(b) Answer: $(6, 2)$

Solution 1

To get from A to B, we move 2 units to the right and 1 unit up.

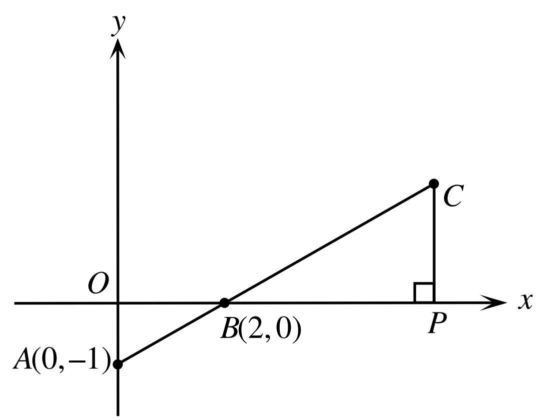


Since C lies on the same straight line as A and B, then to get from B to C we move 2 units to the right and 1 unit up twice, or 4 units to the right and 2 units up.

Thus, the coordinates of C are $(6, 2)$.

Solution 2

Label the origin as O and drop a perpendicular from C to P on the x-axis.



Then $\triangle AOB$ is similar to $\triangle CPB$ since both are right-angled and they have equal angles at B.

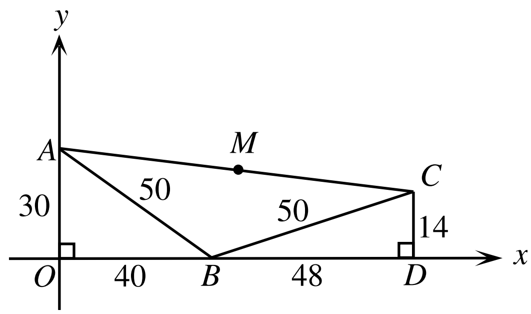
Since $BC = 2AB$, then $CP = 2AO = 2(1) = 2$ and $BP = 2BO = 2(2) = 4$.

Therefore, the coordinates of C are $(2 + 4, 0 + 2) = (6, 2)$.

(c) By the Pythagorean Theorem, $AO^2 = AB^2 - OB^2 = 50^2 - 40^2 = 900$, so $AO = 30$.

Therefore, the coordinates of A are $(0, 30)$.

By the Pythagorean Theorem, $CD^2 = CB^2 - BD^2 = 50^2 - 48^2 = 196$, so $CD = 14$.



Therefore, the coordinates of C are $(40+48,14)=(88,14)$.
Since M is the midpoint of AC, then the coordinates of M are

$$\left(\frac{1}{2}(0+88),\frac{1}{2}(30+14)\right)=(44,22)$$

2. (a) Answer: $x=-2$

Solution 1
Since $y=2x+3$, then $4y=4(2x+3)=8x+12$.
Since $4y=8x+12$ and $4y=5x+6$, then $8x+12=5x+6$ or $3x=-6$ or $x=-2$.

Solution 2
Since $4y=5x+6$, then $y=\frac{5}{4}x+\frac{6}{4}=\frac{5}{4}x+\frac{3}{2}$.
Since $y=2x+3$ and $y=\frac{5}{4}x+\frac{3}{2}$, then $2x+3=\frac{5}{4}x+\frac{3}{2}$ or $\frac{3}{4}x=-\frac{3}{2}$ or $x=-2$.

Solution 3
Since the second equation contains a "$5x$", we multiply the first equation by $\frac{5}{2}$ to obtain a $5x$ term, and obtain $\frac{5}{2}y=5x+\frac{15}{2}$.
Subtracting this from $4y=5x+6$, we obtain $\frac{3}{2}y=-\frac{3}{2}$ or $y=-1$.
Since $y=-1$, then $-1=2x+3$ or $2x=-4$ or $x=-2$.

(b) Answer: $a=6$

Solution 1
Adding the three equations together, we obtain $a-3b+b+2b+7c-2c-5c=-10+3+13$ or $a=6$.

Solution 2
Multiplying the second equation by 3, we obtain $3b-6c=9$.
Adding this new equation to the first equation, we obtain $c=-1$.
Substituting this back into the original second equation, we obtain $b=3+2c=1$.
Substituting into the third equation, $a=-2b+5c+13=-2-5+13=6$.

(c) *Solution 1*
Let J be John's score and M be Mary's score.
Since two times John's score was 60 more than Mary's score, then $2J=M+60$.
Since two times Mary's score was 90 more than John's score, then $2M=J+90$.
Adding these two equations, we obtain $2J+2M=M+J+150$ or $J+M=150$ or $\dfrac{J+M}{2}=75$.
Therefore, the average of their two scores was 75.
(Note that we didn't have to solve for their individual scores.)

Solution 2
Let J be John's score and M be Mary's score.
Since two times John's score was 60 more than Mary's score, then $2J = M + 60$, so $M = 2J - 60$.
Since two times Mary's score was 90 more than John's score, then $2M = J + 90$.
Substituting the first equation into the second, we obtain

$$\begin{aligned} 2(2J - 60) &= J + 90 \\ 4J - 120 &= J + 90 \\ 3J &= 210 \\ J &= 70 \end{aligned}$$

Substituting into $M = 2J - 60$ gives $M = 80$.
Therefore, the average of their scores (ie. the average of 70 and 80) is 75.

3. (a) ANSWER: $x = 50$
Simplifying using exponent rules,

$$2(16^{12}) + 2(8^{16}) = 2((2^4)^{12}) + 2((2^3)^{16}) = 2(2^{48}) + 2(2^{48}) = 4(2^{48}) = 2^2(2^{48}) = 2^{50}$$

Therefore, since $2^x = 2(16^{12}) + 2(8^{16}) = 2^{50}$, then $x = 50$.

(b) *Solution 1*
We factor the given equation $(f(x))^2 - 3f(x) + 2 = 0$ as $(f(x) - 1)(f(x) - 2) = 0$.
Therefore, $f(x) = 1$ or $f(x) = 2$.
If $f(x) = 1$, then $2x - 1 = 1$ or $2x = 2$ or $x = 1$.
If $f(x) = 2$, then $2x - 1 = 2$ or $2x = 3$ or $x = \frac{3}{2}$.
Therefore, the values of x are $x = 1$ or $x = \frac{3}{2}$.

Solution 2
Since $f(x) = 2x - 1$ and $(f(x))^2 - 3f(x) + 2 = 0$, then

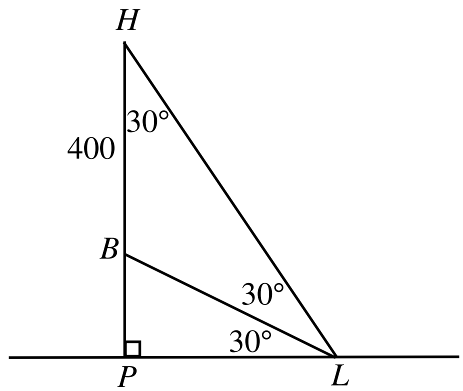
$$\begin{aligned} (2x-1)^2 - 3(2x-1) + 2 &= 0 \\ 4x^2 - 4x + 1 - 6x + 3 + 2 &= 0 \\ 4x^2 - 10x + 6 &= 0 \\ 2x^2 - 5x + 3 &= 0 \\ (x-1)(2x-3) &= 0 \end{aligned}$$

Therfore, $x = 1$ or $x = \frac{3}{2}$.

4. (a) ANSWER: $\frac{14}{15}$
Solution 1
The possible pairs of numbers on the tickets are (listed as ordered pairs): $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,5)$, $(4,6)$, and $(5,6)$.
There are fifteen such pairs. (We treat the pair of tickets numbered 2 and 4 as being the same as the pair numbered 4 and 2.)
The pairs for which the smaller of the two numbers is less than or equal to 4 are $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,3)$, $(2,4)$, $(2,5)$, $(2,6)$, $(3,4)$, $(3,5)$, $(3,6)$, $(4,5)$, and $(4,6)$.
There are fourteen such pairs.
Therefore, the probability of selecting such a pair of tickets is $\frac{14}{15}$.

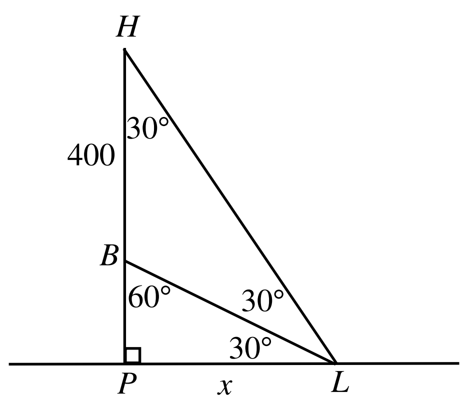
Solution 2
We find the probability that the smaller number on the two tickets is NOT less than or equal to 4.
Therefore, the smaller number on the two tickets is at least 5.
Thus, the pair of numbers must be 5 and 6, since two distinct numbers less than or equal to 6 are being chosen.
As in Solution 1, we can determine that there are fifteen possible pairs that we can selected.
Therefore, the probability that the smaller number on the two tickets is NOT less than or equal to 4 is $\frac{1}{15}$, so the probability that the smaller number on the two tickets IS less than or equal to 4 is $1 - \frac{1}{15} = \frac{14}{15}$.

(b) *Solution 1*
Since $\angle HLP = 60^\circ$ and $\angle BLP = 30^\circ$, then $\angle HLB = \angle HLP - \angle BLP = 30^\circ$.
Also, since $\angle HLP = 60^\circ$ and $\angle HPL = 90^\circ$, then $\angle LHP = 180^\circ - 90^\circ - 60^\circ = 30^\circ$.



Therefore, $\triangle HBL$ is isosceles and $BL = HB = 400$ m.
In $\triangle BLP$, $BL = 400$ m and $\angle BLP = 30^\circ$, so $LP = BL\cos(30^\circ) = 400\left(\frac{\sqrt{3}}{2}\right) = 200\sqrt{3}$ m.
Therefore, the distance between L and P is $200\sqrt{3}$ m.

Solution 2
Since $\angle HLP = 60^\circ$ and $\angle BLP = 30^\circ$, then $\angle HLB = \angle HLP - \angle BLP = 30^\circ$.
Also, since $\angle HLP = 60^\circ$ and $\angle HPL = 90^\circ$, then $\angle LHP = 180^\circ - 90^\circ - 60^\circ = 30^\circ$.
Also, $\angle LBP = 60^\circ$.
Let $LP = x$.



Since $\triangle BLP$ is 30°-60°-90°, then $BP : LP = 1 : \sqrt{3}$, so $BP = \frac{1}{\sqrt{3}}LP = \frac{1}{\sqrt{3}}x$.

Since $\triangle HLP$ is 30°-60°-90°, then $HP : LP = \sqrt{3} : 1$, so $HP = \sqrt{3}LP = \sqrt{3}x$.
But $HP = HB + BP$ so

$$\begin{aligned}
\sqrt{3}x &= 400 + \frac{1}{\sqrt{3}}x \\
3x &= 400\sqrt{3} + x \\
2x &= 400\sqrt{3} \\
x &= 200\sqrt{3}
\end{aligned}$$

Therefore, the distance from L to P is $200\sqrt{3}$ m.

5. (a) Answer: (6,5)
After 2 moves, the goat has travelled $1 + 2 = 3$ units.
After 3 moves, the goat has travelled $1 + 2 + 3 = 6$ units.
Similarly, after n moves, the goat has travelled a total of $1 + 2 + 3 + \cdots + n$ units.
For what value of n is $1 + 2 + 3 + \cdots + n$ equal to 55?
The fastest way to determine the value of n is by adding the first few integers until we obtain a sum of 55. This will be $n = 10$.
(We could also do this by remembering that $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n + 1)$ and solving for n this way.)
So we must determine the coordinates of the goat after 10 moves.
We consider first the x-coordinate.
Since starting at $(0, 0)$ the goat has moved 2 units in the positive x direction, 4 units in the negative x direction, 6 units in the positive x direction, 8 units in the negative x direction and 10 units in the positive x direction, so its x coordinate should be $2-4+6-8+10 = 6$.
Similarly, its y-coordinate should be $1 - 3 + 5 - 7 + 9 = 5$.
Therefore, after having travelled a distance of 55 units, the goat is at the point $(6, 5)$.

(b) *Solution 1*
Since the sequence 4, $4r$, $4r^2$ is also arithmetic, then the difference between $4r^2$ and $4r$ equals the difference between $4r$ and 4, or

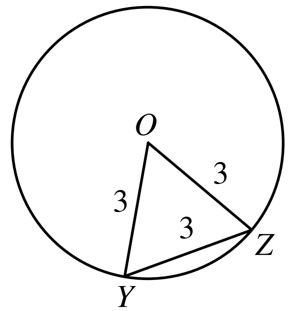
$$\begin{aligned}
4r^2 - 4r &= 4r - 4 \\
4r^2 - 8r + 4 &= 0 \\
r^2 - 2r + 1 &= 0 \\
(r - 1)^2 &= 0
\end{aligned}$$

Therefore, the only value of r is $r = 1$.

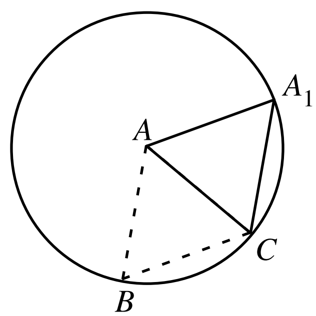
Solution 2
Since the sequence 4, $4r$, $4r^2$ is also arithmetic, then we can write $4r = 4 + d$ and $4r^2 = 4 + 2d$ for some real number d. (Here, d is the common difference in this arithmetic sequence.)
Then $d = 4r - 4$ and $2d = 4r^2 - 4$ or $d = 2r^2 - 2$.
Therefore, equating the two expressions for d, we obtain $2r^2-2 = 4r-4$ or $2r^2-4r+2 = 0$ or $r^2 - 2r + 1 = 0$ or $(r - 1)^2 = 0$.
Therefore, the only value of r is $r = 1$.

6. (a) Answer: 4π
First, we notice that whenever an equilateral triangle of side length 3 is placed inside a

circle of radius 3 with two of its vertices on the circle, then the third vertex will be at the centre of the circle.
This is because if we place $\triangle XYZ$ with Y and Z on the circle and connect Y and Z to the centre O, then $OY = OZ = 3$, so $\triangle OYZ$ is equilateral (since all three sides have length 3). Thus $\triangle XYZ$ and $\triangle OYZ$ must be the same, so X is at the same point as O.

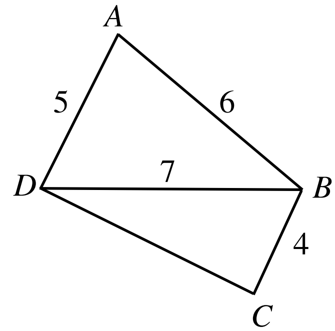


Thus, in the starting position, A is at the centre of the circle.
As the triangle is rotated about C, the point B traces out an arc of a circle of radius 3.
What fraction of the circle is traced out?
When point A reaches point A_1 on the circle, we have $AC = 3$ and $CA_1 = 3$. Since A is at the centre of the circle, then $AA_1 = 3$ as well, so $\triangle AA_1C$ is equilateral, and $\angle A_1CA = 60^\circ$, so the triangle has rotated through 60°.



Therefore, B has traced out $\frac{60^\circ}{360^\circ} = \frac{1}{6}$ of a circle of radius 3.
Notice that A has also traced out an arc of the same length. When A reaches the circle, we have A and C on the circle, so B must be at the centre of the circle.
Thus, on the next rotation, B again rotates through $\frac{1}{6}$ of a circle of radius 3 as it moves to the circle.
On the third rotation, the triangle rotates about B, so B does not move. After three rotations, the triangle will have A at the centre and B and C on the circle, with the net result that the triangle has rotated 180° about the centre of the circle.
Thus, to return to its original position, the triangle must undergo three more of these rotations, and B will behave in the same way as it did for the first three rotations.
Thus, in total, B moves four times along an arc equal to $\frac{1}{6}$ of a circle of radius 3.
Therefore, the distance travelled by B is $4(\frac{1}{6})(2\pi(3)) = 4\pi$.

(b) In order to determine CD, we must determine one of the angles (or at least some information about one of the angles) in $\triangle BCD$.
To do this, we look at $\angle A$ use the fact that $\angle A + \angle C = 180^\circ$.



Using the cosine law in $\triangle ABD$, we obtain

$$\begin{aligned} 7^2 &= 5^2 + 6^2 - 2(5)(6)\cos(\angle A) \\ 49 &= 61 - 60\cos(\angle A) \\ \cos(\angle A) &= \frac{1}{5} \end{aligned}$$

Since $\cos(\angle A) = \frac{1}{5}$ and $\angle A + \angle C = 180^\circ$, then $\cos(\angle C) = -\cos(180^\circ - \angle A) = -\frac{1}{5}$.
(We could have calculated the actual size of $\angle A$ using $\cos(\angle A) = \frac{1}{5}$ and then used this to calculate the size of $\angle C$, but we would introduce the possibility of rounding error by doing this.)
Then, using the cosine law in $\triangle BCD$, we obtain

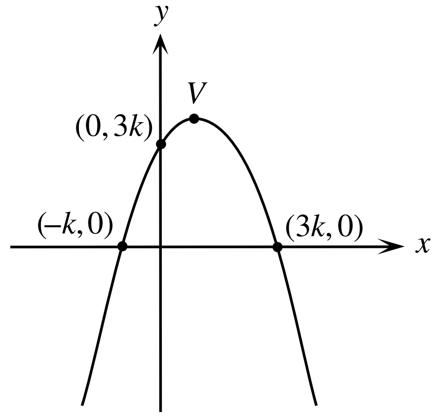
$$\begin{aligned} 7^2 &= 4^2 + CD^2 - 2(4)(CD)\cos(\angle C) \\ 49 &= 16 + CD^2 - 8(CD)\left(-\frac{1}{5}\right) \\ 0 &= 5CD^2 + 8CD - 165 \\ 0 &= (5CD + 33)(CD - 5) \end{aligned}$$

So $CD = -\frac{33}{5}$ or $CD = 5$. (We could have also determined these roots using the quadratic formula.)
Since CD is a length, it must be positive, so $CD = 5$.

(We could have also proceeded by using the sine law in $\triangle BCD$ to determine $\angle BDC$ and then found the size of $\angle DBC$, which would have allowed us to calculate CD using the sine law. However, this would again introduce the potential of rounding error.)

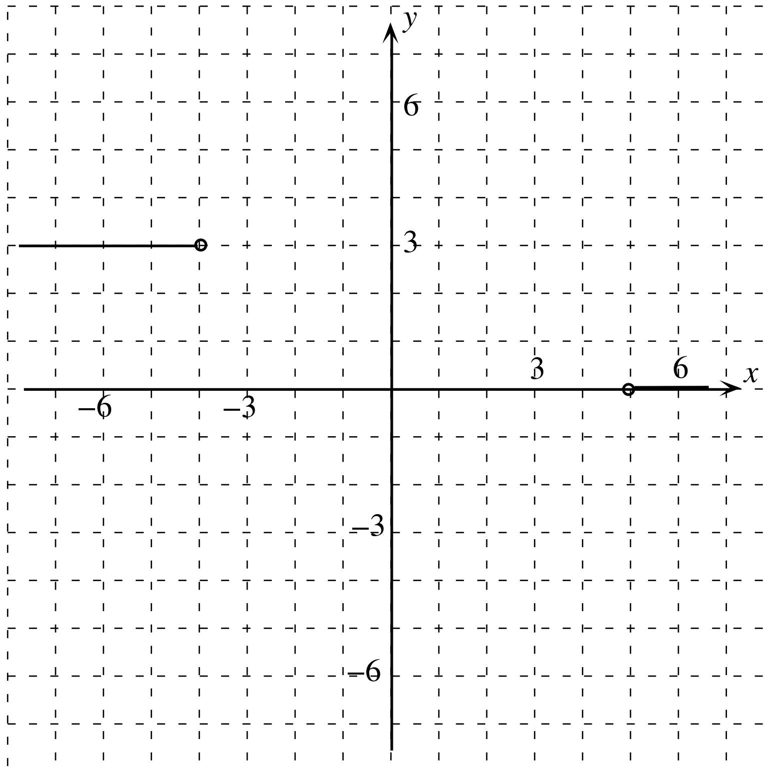
7. (a) Answer: Maximum = 5, Minimum = 1
We rewrite by completing the square as $f(x) = \sin^2 x - 2\sin x + 2 = (\sin x - 1)^2 + 1$.
Therefore, since $(\sin x - 1)^2 \geq 0$, then $f(x) \geq 1$, and in fact $f(x) = 1$ when $\sin x = 1$ (which occurs for instance when $x = 90^\circ$).
Thus, the minimum value of $f(x)$ is 1.
To maximize $f(x)$, we must maximize $(\sin x - 1)^2$.
Since $-1 \leq \sin x \leq 1$, then $(\sin x - 1)^2$ is maximized when $\sin x = -1$ (for instance, when $x = 270^\circ$). In this case, $(\sin x - 1)^2 = 4$, so $f(x) = 5$.
Thus, the maximum value of $f(x)$ is 5.

(b) From the diagram, the x-intercepts of the parabola are $x = -k$ and $x = 3k$.

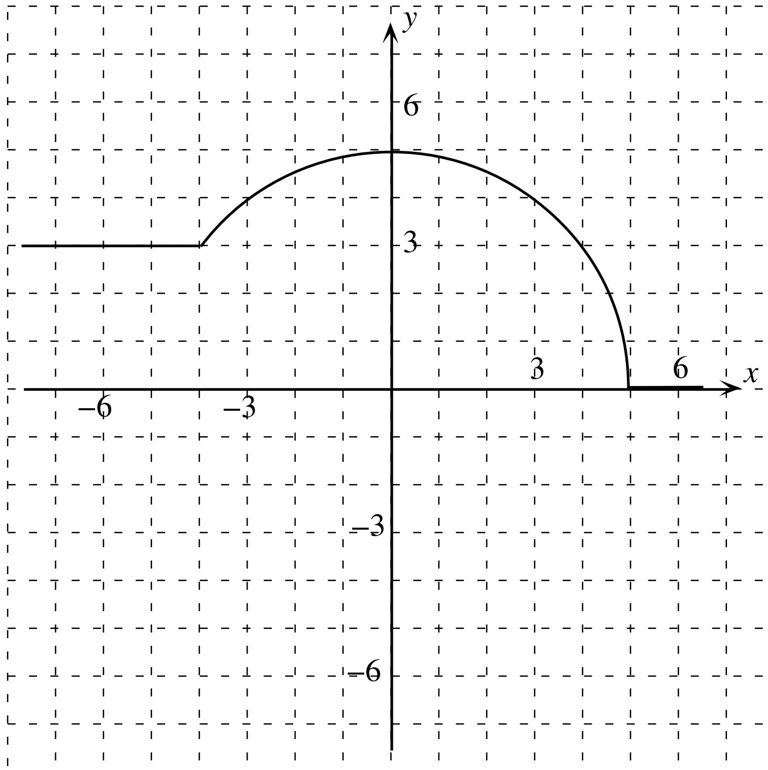


Since we are given that $y = -\frac{1}{4}(x-r)(x-s)$, then the x-intercepts are r and s, so r and s equal $-k$ and $3k$ in some order.
Therefore, we can rewrite the parabola as $y = -\frac{1}{4}(x-(-k))(x-3k)$.
Since the point $(0,3k)$ lies on the parabola, then $3k = -\frac{1}{4}(0+k)(0-3k)$ or $12k = 3k^2$ or $k^2 - 4k = 0$ or $k(k-4) = 0$.
Thus, $k = 0$ or $k = 4$.
Since the two roots are distinct, then we cannot have $k = 0$ (otherwise both x-intercepts would be 0).
Thus, $k = 4$.
This tells us that the equation of the parabola is $y = -\frac{1}{4}(x+4)(x-12)$ or $y = -\frac{1}{4}x^2 + 2x + 12$.
We still have to determine the coordinates of the vertex, V.
Since the x-intercepts of the parabola are -4 and 12, then the x-coordinate of the vertex is the average of these intercepts, or 4.
(We could have also used the fact that the x-coordinate is $-\dfrac{b}{2a} = -\dfrac{2}{2(-\frac{1}{4})}$.)
Therefore, the y-coordinate of the vertex is $y = -\frac{1}{4}(4^2) + 2(4) + 12 = 16$.
Thus, the coordinates of the vertex are $(4,16)$.

8. (a) We look at the three pieces separately.
If $x < -4$, $f(x) = 4$ so $g(x) = \sqrt{25 - [f(x)]^2} = \sqrt{25 - 4^2} = \sqrt{9} = 3$.
So $g(x)$ is the horizontal line $y = 3$ when $x < -4$.
If $x > 5$, $f(x) = -5$ so $g(x) = \sqrt{25 - [f(x)]^2} = \sqrt{25 - (-5)^2} = \sqrt{0} = 0$.
So $g(x)$ is the horizontal line $y = 0$ when $x > 5$.
So far, our graph looks like this:

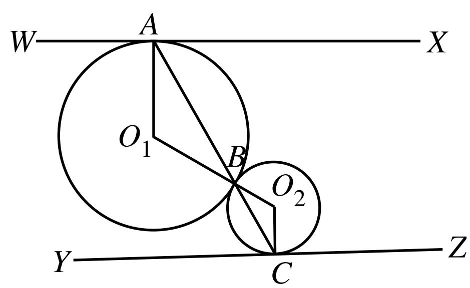


If $-4 \le x \le 5$, $f(x) = -x$ so $g(x) = \sqrt{25 - [f(x)]^2} = \sqrt{25 - (-x)^2} = \sqrt{25 - x^2}$.
What is this shape?
If $y = g(x)$, then we have $y = \sqrt{25 - x^2}$ or $y^2 = 25 - x^2$ or $x^2 + y^2 = 25$.
Therefore, this shape is a section of the upper half (since y is a positive square-root) of the circle $x^2 + y^2 = 25$, ie. the circle with centre $(0, 0)$ and radius 5.
We must check the endpoints.
When $x = -4$, we have $g(-4) = \sqrt{25 - (-4))^2} = 3$.
When $x = 5$, we have $g(5) = \sqrt{25 - 5^2} = 0$.
Therefore, the section of the circle connects up with the other two sections of our graph already in place.
Thus, our final graph is:



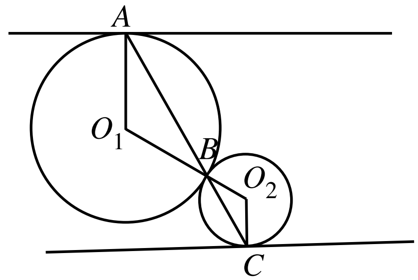
(b) *Solution 1*

Let the centres of the two circles be O_1 and O_2.
Join A and B to O_1 and B and C to O_2.
Designate two points W and X on either side of A on one tangent line, and two points Y and Z on either side of C on the other tangent line.



Let $\angle XAB = \theta$.
Since WX is tangent to the circle with centre O_1 at A, then O_1A is perpendicular to WX, so $\angle O_1AB = 90^\circ - \theta$.
Since $O_1A = O_1B$ because both are radii, then $\triangle AO_1B$ is isosceles, so $\angle O_1BA = \angle O_1AB = 90^\circ - \theta$.
Since the two circles are tangent at B, then the line segment joining O_1 and O_2 passes through B, ie. O_1BO_2 is a straight line segment.
Thus, $\angle O_2BC = \angle O_1BA = 90^\circ - \theta$, by opposite angles.
Since $O_2B = O_2C$, then similarly to above, $\angle O_2CB = \angle O_2BC = 90^\circ - \theta$.
Since YZ is tangent to the circle with centre O_2 at C, then O_2C is perpendicular to YZ.
Thus, $\angle YCB = 90^\circ - \angle O_2CB = \theta$.
Since $\angle XAB = \angle YCB$, then WX is parallel to YZ, by alternate angles, as required.

Solution 2
Let the centres of the two circles be O_1 and O_2.
Join A and B to O_1 and B and C to O_2.
Since AO_1 and BO_1 are radii of the same circle, $AO_1 = BO_1$ so $\triangle AO_1B$ is isosceles, so $\angle O_1AB = \angle O_1BA$.



Since BO_2 and CO_2 are radii of the same circle, $BO_2 = CO_2$ so $\triangle BO_2C$ is isosceles, so $\angle O_2BC = \angle O_2CB$.
Since the two circles are tangent at B, then O_1BO_2 is a line segment (ie. the line segment joining O_1 and O_2 passes through the point of tangency of the two circles).
Since O_1BO_2 is straight, then $\angle O_1BA = \angle O_2BC$, by opposite angles.
Thus, $\angle O_1AB = \angle O_1BA = \angle O_2BC = \angle O_2CB$.
This tells us that $\triangle AO_1B$ is similar to $\triangle BO_2C$, so $\angle AO_1B = \angle BO_2C$ or $\angle AO_1O_2 = \angle CO_2O_1$.
Therefore, AO_1 is parallel to CO_2, by alternate angles.
But A and C are points of tangency, AO_1 is perpendicular to the tangent line at A and CO_2 is perpendicular to the tangent line at C.
Since AO_1 and CO_2 are parallel, then the two tangent lines must be parallel.

9. (a) *Solution 1*
We have $(x-p)^2 + y^2 = r^2$ and $x^2 + (y-p)^2 = r^2$, so at the points of intersection,

$$\begin{aligned}(x-p)^2 + y^2 &= x^2 + (y-p)^2 \\ x^2 - 2px + p^2 + y^2 &= x^2 + y^2 - 2py + p^2 \\ -2px &= -2py\end{aligned}$$

and so $x = y$ (since we may assume that $p \neq 0$ otherwise the two circles would coincide).
Therefore, a and b are the two solutions of the equation $(x-p)^2 + x^2 = r^2$ or $2x^2 - 2px + (p^2 - r^2) = 0$ or $x^2 - px + \frac{1}{2}(p^2 - r^2)$=0.
Using the relationship between the sum and product of roots of a quadratic equation and its coefficients, we obtain that $a + b = p$ and $ab = \frac{1}{2}(p^2 - r^2)$.
(We could have solved for a and b using the quadratic formula and calculated these directly.)
So we know that $a + b = p$.
Lastly, $a^2 + b^2 = (a+b)^2 - 2ab = p^2 - 2\left(\frac{1}{2}(p^2 - r^2)\right) = r^2$, as required.

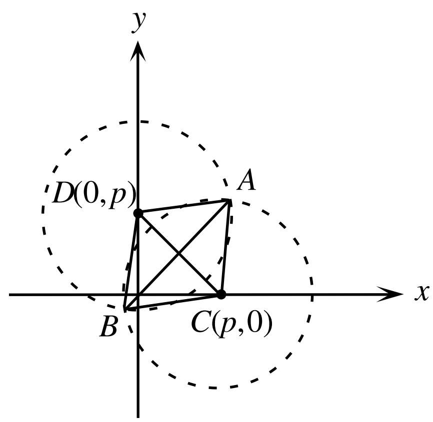
Solution 2
Since the circles are reflections of one another in the line $y = x$, then the two points of intersection must both lie on the line $y = x$, ie. A has coordinates (a, a) and B has coordinates (b, b).
Therefore, $(a-p)^2 + a^2 = r^2$ and $(b-p)^2 + b^2 = r^2$, since these points lie on both circles.

Subtracting the two equations, we get

$$\begin{aligned}
(b-p)^2-(a-p)^2+b^2-a^2 &= 0\\
((b-p)-(a-p))((b-p)+(a-p))+(b-a)(b+a) &= 0\\
(b-a)(a+b-2p)+(b-a)(b+a) &= 0\\
(b-a)(a+b-2p+b+a) &= 0\\
2(b-a)(a+b-p) &= 0
\end{aligned}$$

Since $a \neq b$, then we must have $a+b=p$, as required.
Since $a+b=p$, then $a-p=-b$, so substituting back into $(a-p)^2+a^2=r^2$ gives $(-b)^2+a^2=r^2$, or $a^2+b^2=r^2$, as required.

(b) We first draw a diagram.



We know that C has coordinates $(p,0)$ and D has coordinates $(0,p)$.
Thus, the slope of line segment CD is -1.
Since the points A and B both lie on the line $y=x$, then the slope of line segment AB is 1.
Therefore, AB is perpendicular to CD, so $CADB$ is a kite, and so its area is equal to $\frac{1}{2}(AB)(CD)$.
(We could derive this by breaking quadrilateral $CADB$ into $\triangle CAB$ and $\triangle DAB$.)
Since C has coordinates $(p,0)$ and D has coordinates $(0,p)$, then $CD=\sqrt{p^2+(-p)^2}=\sqrt{2p^2}$.
(We do not know if p is positive, so this is not necessarily equal to $\sqrt{2}p$.)
We know that A has coordinates (a,a) and B has coordinates (b,b), so

$$\begin{aligned}
AB &= \sqrt{(a-b)^2+(a-b)^2}\\
&= \sqrt{2a^2-4ab+2b^2}\\
&= \sqrt{2(a^2+b^2)-4ab}\\
&= \sqrt{2r^2-4\left(\frac{1}{2}(p^2-r^2)\right)}\\
&= \sqrt{4r^2-2p^2}
\end{aligned}$$

Therefore, the area of quadrilateral $CADB$ is

$$\frac{1}{2}(AB)(CD)=\frac{1}{2}\sqrt{4r^2-2p^2}\sqrt{2p^2}=\sqrt{2r^2p^2-p^4}$$

To maximize this area, we must maximize $2r^2p^2 - p^4 = 2r^2(p^2) - (p^2)^2$.
Since r is fixed, we can consider this as a quadratic polynomial in p^2. Since the coefficient of $(p^2)^2$ is negative, then this is a parabola opening downwards, so we find its maximum value by finding its vertex.
The vertex of $2r^2(p^2) - (p^2)^2$ is at $p^2 = -\dfrac{2r^2}{2(-1)} = r^2$.
So the maximum area of the quadrilateral occurs when p is chosen so that $p^2 = r^2$.
Since $p^2 = r^2$, then $(a+b)^2 = p^2 = r^2$ so $a^2 + 2ab + b^2 = r^2$.
Since $a^2 + b^2 = r^2$, then $2ab = 0$ so either $a = 0$ or $b = 0$, and so either A has coordinates $(0,0)$ or B has coordinates $(0,0)$, ie. either A is the origin or B is the origin.

(c) In (b), we calculated that $AB = \sqrt{4r^2 - 2p^2} = \sqrt{2}\sqrt{2r^2 - p^2}$.
Since r and p are integers (and we assume that neither r nor p is 0), then $2r^2 - p^2 \neq 0$, so the minimum possible non-negative value for $2r^2 - p^2$ is 1, since $2r^2 - p^2$ must be an integer.
Therefore, the minimum possible distance between A and B should be $\sqrt{2}\sqrt{1} = \sqrt{2}$.
Can we find positive integers p and r that give us this value?
Yes – if $r = 5$ and $p = 7$, then $2r^2 - p^2 = 1$, so $AB = \sqrt{2}$.
(There are in fact an infinite number of positive integer solutions to the equation $2r^2 - p^2 = 1$ or equivalently $p^2 - 2r^2 = -1$. This type of equation is called Pell's Equation.)

10. (a) We proceed directly.
On the first pass from left to right, Josephine closes all of the even numbered lockers, leaving the odd ones open.
The second pass proceeds from right to left. Before the pass, the lockers which are open are 1, 3, ..., 47, 49.
On the second pass, she shuts lockers 47, 43, 39, ..., 3.
The third pass proceeds from left to right. Before the pass, the lockers which are open are 1, 5, ..., 45, 49.
On the third pass, she shuts lockers 5, 13, ..., 45.
This leaves lockers 1, 9, 17, 25, 33, 41, 49 open.
On the fourth pass, from right to left, lockers 41, 25 and 9 are shut, leaving 1, 17, 33, 49.
On the fifth pass, from left to right, lockers 17 and 49 are shut, leaving 1 and 33 open.
On the sixth pass, from right to left, locker 1 is shut, leaving 33 open.
Thus, $f(50) = 33$.

(b)&(c) *Solution 1*
First, we note that if $n = 2k$ is even, then $f(n) = f(2k) = f(2k-1) = f(n-1)$. See Solution 2 for this justification.
Therefore, we only need to look for odd values of n in parts (b) and (c).

Suppose that there was an n so that $f(n) = 2005$, ie. 2005 is the last locker left open.
On the first pass, Josephine closes every other locker starting at the beginning, so she closes all lockers numbered m with $m \equiv 0 \pmod{2}$.
This leaves only odd-numbered lockers open, ie. only lockers m with $m \equiv 1$ or $3 \pmod{4}$.
On her second pass, she closes every other open locker, starting from the right-hand end. Thus, she will close every fourth locker from the original row.
Since we want 2005 to be left open and $2005 \equiv 1 \pmod{4}$, then she must close all lockers numbered m with $m \equiv 3 \pmod{4}$.
This leaves open only the lockers m with $m \equiv 1 \pmod{4}$, or equivalently lockers with $m \equiv 1$ or $5 \pmod{8}$.

On her third pass, she closes every other open locker, starting from the left-hand end. Thus, she will close every eighth locker from the original row.
Since locker 1 is still open, then she starts by closing locker 5, and so closes all lockers m with $m \equiv 5 \pmod 8$.
But since $2005 \equiv 5 \pmod 8$, then she closes locker 2005 on this pass, a contradiction.
Therefore, there can be no integer n with $f(n) = 2005$.

Next, we show that there are infinitely many positive integers n such that $f(n) = f(2005)$. To do this, we first make a table of what happens when there are 2005 lockers in the row. We record the pass #, the direction of the pass, the leftmost locker that is open, the rightmost locker that is open, all open lockers before the pass, which lockers will be closed on the pass, and which lockers will be left open after the pass:

Pass #	Dir.	L Open	R Open	Open	To close	Leaves Open
1	L to R	1	2005	All	$\equiv 0 \pmod 2$	$\equiv 1 \pmod 2$
2	R to L	1	2005	$\equiv 1, 3 \pmod 4$	$\equiv 3 \pmod 4$	$\equiv 1 \pmod 4$
3	L to R	1	2005	$\equiv 1, 5 \pmod 8$	$\equiv 5 \pmod 8$	$\equiv 1 \pmod 8$
4	R to L	1	2001	$\equiv 1, 9 \pmod{16}$	$\equiv 9 \pmod{16}$	$\equiv 1 \pmod{16}$
5	L to R	1	2001	$\equiv 1, 17 \pmod{32}$	$\equiv 17 \pmod{32}$	$\equiv 1 \pmod{32}$
6	R to L	1	1985	$\equiv 1, 33 \pmod{64}$	$\equiv 33 \pmod{64}$	$\equiv 1 \pmod{64}$
7	L to R	1	1985	$\equiv 1, 65 \pmod{128}$	$\equiv 65 \pmod{128}$	$\equiv 1 \pmod{128}$
8	R to L	1	1921	$\equiv 1, 129 \pmod{256}$	$\equiv 1 \pmod{256}$	$\equiv 129 \pmod{256}$
9	L to R	129	1921	$\equiv 129, 385 \pmod{512}$	$\equiv 385 \pmod{512}$	$\equiv 129 \pmod{512}$
10	R to L	129	1665	$\equiv 129, 641 \pmod{1024}$	$\equiv 129 \pmod{1024}$	$\equiv 641 \pmod{1024}$
11	L to R	641	1665	$\equiv 641, 1665 \pmod{2048}$	$\equiv 1665 \pmod{2048}$	$\equiv 641 \pmod{2048}$

Since there is only one integer between 1 and 2005 congruent to $641 \pmod{2048}$, then there is only one locker left open: locker 641.
Notice also that on any pass s, the "class" of lockers which are closed depends on what the number of the leftmost (on an odd-numbered pass) or rightmost (on an even-numbered pass) open locker number is congruent to mod 2^s.

Consider $n = 2005 + 2^{2a}$, where $2^{2a} > 2005$, ie. $a \geq 6$.
We show that $f(n) = f(2005) = 641$. (See Solution 2 for a justification of why we might try these values of n.)
Suppose we were to try to make a table as above to calculate $f(n)$.
Then the first 11 passes in the table would be identical to the table above, except for the rightmost open number; this number in the new table would be the number above plus 2^{2a}.
What will happen after pass 11?
After pass 11, the lockers which are open are lockers with numbers $\equiv 641 \pmod{2048}$. Thus, the leftmost open locker is 641 and the rightmost is $2^{2a} + 641$.
As the 12th pass starts, the lockers which are still open are those with numbers $\equiv 641$ or $2689 \pmod{2^{12}}$.
Since the rightmost open locker number $(2^{2a}+641)$ is congruent to $641 \pmod{2^{12}}$, then all lockers with numbers $\equiv 2689 \pmod{2^{12}}$ are closed, leaving open only those lockers with numbers $\equiv 641 \pmod{2^{12}}$.
So after this 12th pass, the lockers which are open are 641, $641 + 2^{12}$, $641 + 2(2^{12})$, $641 + 3(2^{12})$, $\ldots$, $641 + 2^{2a-12}(2^{12}) = 641 + 2^{2a}$.
The number of open lockers is $2^{2a-12} + 1$.

If we can now show that whenever we start with a number of lockers of the form $2^{2c}+1$, the last locker remaining open is the leftmost locker, then we will be done, since of the lockers left open above ($2^{2a-12}+1$ of them, ie. 2 to an even power plus 1), then the last locker remaining open will be the leftmost one, that is locker 641, so $f(2^{2a}+2005) = 641 = f(2005)$.

So consider a row of $2^{2c}+1$ lockers.
Notice that on any pass, if the number of lockers is odd, then the number of lockers which will be closed is one-half of one less than the total number of lockers, and the first and last lockers will be left open.
So on the first pass, there are 2^{2c-1} lockers closed, leaving $2^{2c}+1-2^{2c-1} = 2^{2c-1}+1$ lockers open, ie. an odd number of lockers open.
On the next pass, there are 2^{2c-2} lockers closed (since there are an odd number of lockers open to begin), leaving $2^{2c-2}+1$ lockers open.
This continues, until there are $2^1+1=3$ lockers open just before an even-numbered (ie. right to left) pass. Thus, the middle of these three lockers will be closed, leaving only the original leftmost and rightmost lockers open.
On the last pass (an odd-numbered pass from left to right), the rightmost locker will be closed, leaving only the leftmost locker open.
Therefore, starting with a row of $2^{2c}+1$ open lockers, the leftmost locker will be the last remaining open.

Translating this to the above, we see that the leftmost locker of the $2^{2a-12}+1$ still open is the last left open, ie. $f(2^{2a}+2005) = 641 = f(2005)$ if $a \geq 6$.

Therefore, there are infinitely many positive integers n for which $f(n) = f(2005)$.

Solution 2

First, we calculate $f(n)$ for n from 1 to 32, to get a feeling for what happens. We obtain 1, 1, 3, 3, 1, 1, 3, 3, 9, 9, 11, 11, 9, 9, 11, 11, 1, 1, 3, 3, 1, 1, 3, 3, 9, 9, 11, 11, 9, 9, 11, 11. This will help us to establish some patterns.

Next, we establish two recursive formulas for $f(n)$.

First, from our pattern, it looks like $f(2m) = f(2m-1)$.
Why is this true in general?
Consider a row of $2m$ lockers.
On the first pass through , Josephine shuts all of the even numbered lockers, leaving open lockers 1, 3, …, $2m-1$.
These are exactly the same open lockers as if she had started with $2m-1$ lockers in total.
Thus, as she starts her second pass from right to left, the process will be the same now whether she started with $2m$ lockers or $2m-1$ lockers.
Therefore, $f(2m) = f(2m-1)$.
This tells us that we need only focus on the values of $f(n)$ where n is odd.

Secondly, we show that $f(2m-1) = 2m+1-2f(m)$.
(It is helpful to connect $n = 2m-1$ to a smaller case.)
Why is this formula true?
Starting with $2m-1$ lockers, the lockers left open after the first pass are 1, 3, …, $2m-1$, ie. m lockers in total.
Suppose $f(m) = p$. As Josephine begins her second pass, which is from right to left, we can think of this as being like the first pass through a row of m lockers.
Thus, the last open locker will be the pth locker, counting from the right hand end, from the list 1, 3, …, $2m-1$.
The first locker from the right is $2m-1 = 2m+1-2(1)$, the second is $2m-3 = 2m+1-2(2)$, and so on, so the pth locker is $2m+1-2p$.
Therefore, the final open locker is $2m+1-2p$, ie. $f(2m-1) = 2m+1-2p = 2m+1-2f(m)$.

Using these two formulae repeatedly,

$$\begin{aligned}
f(4k+1) &= f(2(2k+1)-1) \\
&= 2(2k+1)+1-2f(2k+1) \\
&= 4k+3-2f(2(k+1)-1) \\
&= 4k+3-2(2(k+1)+1-2f(k+1)) \\
&= 4k+3-2(2k+3-2f(k+1)) \\
&= 4f(k+1)-3
\end{aligned}$$

and

$$\begin{aligned} f(4k+3) &= f(2(2k+2)-1) \\ &= 2(2k+2)+1-2f(2k+2) \\ &= 4k+5-2f(2k+1) \\ &= 4k+5-2f(2(k+1)-1) \\ &= 4k+5-2(2(k+1)+1-2f(k+1)) \\ &= 4k+5-2(2k+3-2f(k+1)) \\ &= 4f(k+1)-1 \end{aligned}$$

From our initial list of values of $f(n)$, it appears as if $f(n)$ cannot leave a remainder of 5 or 7 when divided by 8. So we use these recursive relations once more to try to establish this:

$$\begin{aligned} f(8l+1) &= 4f(2l+1)-3 \qquad (\text{since } 8l+1 = 4(2l)+1) \\ &= 4(2l+3-2f(l+1))-3 \\ &= 8l+9-8f(l+1) \\ &= 8(l-f(l+1))+9 \end{aligned}$$

$$\begin{aligned} f(8l+3) &= 4f(2l+1)-1 \qquad (\text{since } 8l+3 = 4(2l)+3) \\ &= 4(2l+3-2f(l+1))-1 \\ &= 8l+11-8f(l+1) \\ &= 8(l-f(l+1))+11 \end{aligned}$$

Similarly, $f(8l+5) = 8l+9-8f(l+1)$ and $f(8l+7) = 8l+11-8f(l+1)$.
Therefore, since any odd positive integer n can be written as $8l+1$, $8l+3$, $8l+5$ or $8l+7$, then for any odd positive integer n, $f(n)$ is either 9 more or 11 more than a multiple of 8.
Therefore, for any odd positive integer n, $f(n)$ cannot be 2005, since 2005 is not 9 more or 11 more than a multiple of 8.
Thus, for every positive integer n, $f(n) \neq 2005$, since we only need to consider odd values of n.

Next, we show that there are infinitely many positive integers n such that $f(n) = f(2005)$. We do this by looking at the pattern we initially created and conjecturing that

$$f(2005) = f(2005 + 2^{2a})$$

if $2^{2a} > 2005$. (We might guess this by looking at the connection between $f(1)$ and $f(3)$ with $f(5)$ and $f(7)$ and then $f(1)$ through $f(15)$ with $f(17)$ through $f(31)$. In fact, it appears to be true that $f(m + 2^{2a}) = f(m)$ if $2^{2a} > m$.)

Using our formulae from above,

$$\begin{array}{rcll} f(2005+2^{2a}) & = & 4f(502+2^{2a-2})-3 & (2005+2^{2a}=4(501+2^{2a-2})+1) \\ & = & 4f(501+2^{2a-2})-3 & \\ & = & 4(4f(126+2^{2a-4})-3)-3 & (501+2^{2a-2}=4(125+2^{2a-4})+1) \\ & = & 16f(126+2^{2a-4})-15 & \\ & = & 16f(125+2^{2a-4})-15 & \\ & = & 16(4f(32+2^{2a-6})-3)-15 & (125+2^{2a-4}=4(31+2^{2a-6})+1) \\ & = & 64f(32+2^{2a-6})-63 & \\ & = & 64f(31+2^{2a-6})-63 & \\ & = & 64(4f(8+2^{2a-8})-1)-63 & (31+2^{2a-6}=4(7+2^{2a-8})+3) \\ & = & 256f(8+2^{2a-8})-127 & \\ & = & 256f(7+2^{2a-8})-127 & \\ & = & 256(4f(2+2^{2a-10})-1)-127 & (7+2^{2a-8}=4(1+2^{2a-10})+3) \\ & = & 1024f(2+2^{2a-10})-383 & \\ & = & 1024f(1+2^{2a-10})-383 & \end{array}$$

(Notice that we could have removed the powers of 2 from inside the functions and used this same approach to show that $f(2005) = 1024f(1) - 383 = 641$.)

But, $f(2^{2b} + 1) = 1$ for every positive integer b.
Why is this true? We can prove this quickly by induction.
For $b = 1$, we know $f(5) = 1$.
Assume that the result is true for $b = B - 1$, for some positive integer $B \geq 2$.
Then $f(2^{2B} + 1) = f(4(2^{2B-2}) + 1) = 4f(2^{2B-2} + 1) - 3 = 4(1) - 3 = 1$ by our induction hypothesis.

Therefore, if $a \geq 6$, then $f(1 + 2^{2a-10}) = f(1 + 2^{2(a-5)}) = 1$ so

$$f(2005 + 2^{2a}) = 1024(1) - 383 = 641 = f(2005)$$

so there are infinitely many integers n for which $f(n) = f(2005)$.

Solution 3
We conjecture a formula for $f(n)$ and prove this formula by induction, using the formulae that we proved in Solution 2.
We start by writing the positive integer n in its binary representation, ie. we write

$$n = b_0 + b_1 \cdot 2 + b_2 \cdot 2^2 + \cdots + b_{2p-1} \cdot 2^{2p-1} + b_{2p} \cdot 2^{2p}$$

where each of $b_0, b_1, \cdots, b_{2p}$ is 0 or 1 with either $b_{2p} = 1$, or $b_{2p} = 0$ and $b_{2p-1} = 1$.
Thus, in binary, n is equal to either $(b_{2p}b_{2p-1}\cdots b_1b_0)_2$ or $(b_{2p-1}\cdots b_1b_0)_2$.
We then conjecture that if n is odd (which tells us that $b_0 = 1$ for sure), then

$$f(n) = b_0 + b_1 \cdot 2 + b_3 \cdot 2^3 + \cdots + b_{2p-1} \cdot 2^{2p-1}$$

In other words, we omit the even-numbered powers of 2 from n. Looking at a few examples: $7 = 4 + 2 + 1$, so $f(7) = 2 + 1 = 3$, $13 = 8 + 4 + 1$, so $f(13) = 8 + 1 = 9$, and $27 = 16 + 8 + 2 + 1$, so $f(27) = 8 + 2 + 1 = 11$.
We already know that if n is even, then $f(n) = f(n-1)$ (we proved this in Solution 2).

Let's assume that we've proved this formula. (We'll prove it at the end.)

We can now solve parts (b) and (c) very quickly using our formula.
Are then any values of n such that $f(n) = 2005$?
Writing 2005 as a sum of powers of 2 (ie. in binary), we get

$$2005 = 1024 + 512 + 256 + 128 + 64 + 16 + 4 + 1$$

Since the representation of 2005 does not use only odd-numbered powers of 2, then there is no n for which $f(n) = 2005$.

Lastly, we need to prove that there are infinitely many positive integers n for which $f(n) = f(2005)$.
To do this, we note that if $n = 2005 + 2^{2a}$ for some $a \geq 6$, then the last 11 binary digits of n agree with those of 2005 and the only 1s in the representation of $n = 2005 + 2^{2a}$ in positions corresponding to odd-numbered powers of 2 come from the 2005 portion (since the extra "1" from 2^{2a} corresponds to an even-numbered power of 2).
Therefore, since we calculate $f(2005 + 2^{2a})$ by looking at only the odd-numbered powers of 2, then $f(2005 + 2^{2a}) = f(2005)$ for all integers $a \geq 6$.
Therefore, there are infinitely many positive integers n for which $f(n) = f(2005)$.

We now must prove that this formula is true. We use strong induction.
Looking at the list in Solution 2, we can quickly see that the result holds for all odd values of n with $n \leq 31$. (We only need to establish this for a couple of small values of n to serve as base cases.)
Assume that the result holds for all odd positive integers n up to $n = N - 2$ for some odd positive integer N.
Consider $n = N$.
<u>Case 1: $N = 4q + 1$</u>
Here we can write

$$N = 1 + b_2 \cdot 2^2 + \cdots + b_{2p-1} \cdot 2^{2p-1} + b_{2p} \cdot 2^{2p}$$

and so

$$q = b_2 + b_3 \cdot 2 + \cdots + b_{2p-1} \cdot 2^{2p-3} + b_{2p} \cdot 2^{2p-2}$$

Note that $q < N - 2$ since $4q + 1 = N$, so $q = \frac{1}{4}N - \frac{1}{4}$.
From our formulae in Solution 2, $f(N) = f(4q + 1) = 4f(q + 1) - 3$.
If q is even, then $b_2 = 0$ and so $q+1$ is odd and $q+1 = 1+b_3\cdot 2+\cdots+b_{2p-1}\cdot 2^{2p-3}+b_{2p}\cdot 2^{2p-2}$.
If q is odd, then $b_2 = 1$, so $q = 1 + b_3 \cdot 2 + \cdots + b_{2p-1} \cdot 2^{2p-3} + b_{2p} \cdot 2^{2p-2}$ and $q + 1$ is even, so $f(q + 1) = f(q)$.
In either case,

$$f(q+1) = f(1+b_3 \cdot 2+\cdots+b_{2p-1} \cdot 2^{2p-3}+b_{2p} \cdot 2^{2p-2}) = 1+b_3 \cdot 2+b_5 \cdot 2^3+\cdots b_{2p-1} \cdot 2^{2p-3}$$

by our Induction Hypothesis.
Therefore,

$$f(N) = 4(1 + b_3 \cdot 2 + b_5 \cdot 2^3 + \cdots b_{2p-1} \cdot 2^{2p-3}) - 3 = 1 + b_3 \cdot 2^3 + b_5 \cdot 2^5 + \cdots + b_{2p-1} \cdot 2^{2p-1}$$

as we would like, since $b_1 = 0$.

Case 2: $N = 4q + 3$
Here we can write

$$N = 1 + 2 + b_2 \cdot 2^2 + \cdots + b_{2p-1} \cdot 2^{2p-1} + b_{2p} \cdot 2^{2p}$$

and so

$$q = b_2 + b_3 \cdot 2 + \cdots + b_{2p-1} \cdot 2^{2p-3} + b_{2p} \cdot 2^{2p-2}$$

Note that $q < N - 2$ since $4q + 3 = N$.
From our formulae in Solution 2, $f(N) = f(4q + 3) = 4f(q + 1) - 1$.
If q is even, then $b_2 = 0$ and so $q+1$ is odd and $q+1 = 1+b_3\cdot 2+\cdots+b_{2p-1}\cdot 2^{2p-3}+b_{2p}\cdot 2^{2p-2}$.
If q is odd, then $b_2 = 1$, so $q = 1 + b_3 \cdot 2 + \cdots + b_{2p-1} \cdot 2^{2p-3} + b_{2p} \cdot 2^{2p-2}$ and $q + 1$ is even, so $f(q + 1) = f(q)$.
In either case,

$$f(q+1) = f(1+b_3 \cdot 2+\cdots+b_{2p-1} \cdot 2^{2p-3}+b_{2p} \cdot 2^{2p-2}) = 1+b_3 \cdot 2+b_5 \cdot 2^3+\cdots b_{2p-1} \cdot 2^{2p-3}$$

by our Induction Hypothesis.
Therefore,

$$f(N) = 4(1+b_3 \cdot 2+b_5 \cdot 2^3+\cdots b_{2p-1} \cdot 2^{2p-3})-1 = 1+2+\cdot b_3 \cdot 2^3+b_5 \cdot 2^5+\cdots+b_{2p-1} \cdot 2^{2p-1}$$

as we would like.

Therefore, by strong induction, our formula holds. This complete our proof.

Solution 4
First, we note that if $n = 2k$ is even, then $f(n) = f(2k) = f(2k-1) = f(n-1)$. See Solution 2 for this justification.
Therefore, we only need to look for odd values of n in parts (b) and (c).

Write the number n in binary as $n = (b_{2p}b_{2p-1}\cdots b_2b_1 1)_2$, where each digit is either 0 or 1. Here, we allow the possibility of $b_{2p} = 0$ if $b_{2p-1} = 1$. Since n is odd, then the last digit must be 1, as shown in the representation of n.
We conjecture that if $n = (b_{2p}b_{2p-1}\cdots b_2b_1 1)_2$, then $f(n) = (b_{2p-1}0b_{2p-3}0\cdots b_3 0 b_1 1)_2$, ie. we take the binary representation of n and make every digit corresponding to an even power of 2 into a 0.

Assume that we have proven this formula. (We will prove it below.) We can now quickly solve (b) and (c).
Is there an integer n such that $f(n) = 2005$?
Since $2005 = 1024 + 512 + 256 + 128 + 64 + 16 + 4 + 1$, then $2005 = (11111010101)_2$.
Thus, the binary representation of 2005 does not have only 0's in the digits corresponding to even powers of 2, so 2005 cannot be $f(n)$ for any n.

Why are there infinitely many positive integers n for which $f(n) = 2005$?
Consider $n = 2005 + 2^{2a}$ for some positive integer n, where $2^{2a} > 2005$, ie. $n \geq 6$.
Then the binary representation of n is $n = (10\cdots 011111010101)_2$, where the leading 1 is in a digit corresponding to an even power of 2, and so is zeroed when f is applied.
Therefore, $f(n) = (00\cdots 001010000001)_2 = (1010000001)_2 = f(2005)$.
Thus, there are infinitely many positive integers n for which $f(n) = f(2005)$.

Lastly, we must prove that our formula is true.
Write the numbers from 1 to n in binary in a list from top to bottom:

$$
\begin{array}{cccccc}
\ldots & 0 & 0 & 0 & 0 & 1 \\
\ldots & 0 & 0 & 0 & 1 & 0 \\
\ldots & 0 & 0 & 0 & 1 & 1 \\
\ldots & 0 & 0 & 1 & 0 & 0 \\
\ldots & 0 & 0 & 1 & 0 & 1 \\
\ldots & 0 & 0 & 1 & 1 & 0 \\
\ldots & 0 & 0 & 1 & 1 & 1 \\
\ldots & 0 & 1 & 0 & 0 & 0 \\
 & & \vdots & & & \\
\ldots & b_4 & b_3 & b_2 & b_1 & 1
\end{array}
$$

On odd-numbered passes through the lockers, Josephine moves from left to right, corresponding to downwards in this list. On even-numbered passes through the lockers, Josephine moves from right to left, corresponding to upwards in this list.

On the first pass, we remove every other number from this list, moving downwards. Thus, we remove every even number, or all of those $\equiv 0 \pmod 2$, or all of those with 0th binary digit of 0.
Therefore, after the first pass, only those with a 0th binary digit of 1 remain, and the 1st binary digit (ie. corresponding to 2^1) alternates between 0 and 1, since the numbers in the

list alternate between 1 (mod 4) and 3 (mod 4).

On the second pass through the list, which is upwards, we remove every other remaining number. Since the numbers remaining the list alternate between ending in 01 and 11, and we do not remove the last number, then we leave all those numbers ending in $b_1 1$.
(Since we are removing every fourth number from the original list, the final two binary digits of the remaining numbers should all be the same.)
What remains in our list after two passes? The numbers which remain are all congruent to the same thing (an odd number) modulo 4.

Consider the third pass.
Since one out of every four of the original numbers remains and all of the remaining numbers are odd, then the first number still in the list is less than 4.
Since every number remaining in the list is congruent to the same thing modulo 4, then the last three digits alternate $0b_1 1$ and $1b_1 1$ (ie. the last two binary digits are the same).
Since the first number is less than 4, then it ends in $0b_1 1$.
Since we now remove every other number remaining, then we remove all those numbers with last three binary digits $1b_1 1$, leaving only those with last three digits $0b_1 1$. Thus, all remaining numbers are congruent to the same number modulo 8.
What is the last number remaining in the list?
If the original last number in the list was $...b_3 0 b_1 1$ (ie. $b_2 = 0$), then this number still remains.
If the last number before the third pass was $...b_3 1 b_1 1$ (ie. $b_2 = 1$), then the second last remaining number would be $(...b_3 1 b_1 1)_2 - 4 = (...b_3 0 b_1 1)_2$, and it is this second last number which remains. In either case, the last remaining number is $...b_3 0 b_1 1$.

Consider now a general even-numbered pass (say, pass number $2m$ moving through the list from bottom to top).
The last number in the list (ie. the first encountered) will be $...b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$ and the numbers in the list will alternate between ending $...10 b_{2m-3} 0 \cdots b_3 0 b_1 1$ and ending $...00 b_{2m-3} 0 \cdots b_3 0 b_1 1$ (since every 2^{2m-1}th number from the original list remains).
The last number in the list will not be removed, so we will remove all numbers not agreeing with the last number in the $(2m-1)$th digit, ie. we are left with all numbers ending in $...b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$. This leaves us with every 2^{2m}th number from our original list.
Since all remaining numbers are odd, then the smallest number remaining in the list is smaller than 2^{2m}, so ends in $...0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$.

On the next (odd-numbered pass), the list begins with all numbers ending in either
$...0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$ or $...1 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$.
Since the first number encountered ends in $...0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$, then we remove all numbers ending in $...1 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$, leaving only those ending in
$...0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$, ie. every 2^{2m+1}th number from the original list.
Just before this pass, the largest number remaining ended in
$...b_{2m+1} b_{2m} b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$.
After this pass, the largest number remaining will end in $...b_{2m+1} 0 b_{2m-1} 0 b_{2m-3} 0 \cdots b_3 0 b_1 1$, by the same argument we used in the third pass.

Thus, the process continues as expected, and the final number remaining in the list will be $b_{2p-1} 0 b_{2p-3} 0 \cdots b_3 0 b_1 1$, so $f(n) = (b_{2p-1} 0 b_{2p-3} 0 \cdots b_3 0 b_1 1)_2$