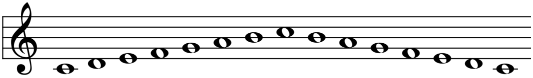
Math in the Real World: Music (9+)

CEMC

The Connection



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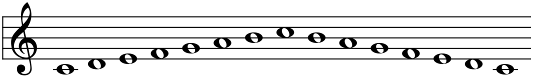

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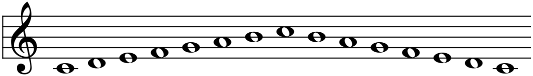

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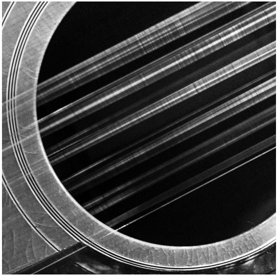
The fundamental mathematics behind these scales? Fractions!

Frequency



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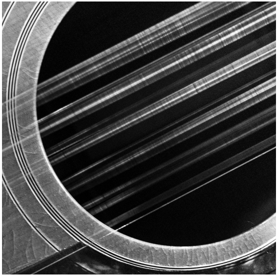

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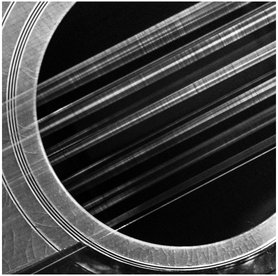

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This is measured in a unit called the Hertz (*Hz*). One Hertz means one vibration per second.

An Introduction to Scales

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Thus, the 'Pythagorean Scale' was created.

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This leaves us with four missing notes in our eight note scale.

To fill in the last four, we iterate by multiplying by $\frac{3}{2}$, and every time we obtain a value *greater than* 2, we halve it. We do this because we do not want any notes in our scale higher than an octave above the base. **Try it out!**

Pythagorean Scale

We get the values below:

$$1 \qquad \frac{3}{2} \qquad \frac{9}{8} \qquad \frac{27}{16} \qquad \frac{81}{64} \qquad \frac{243}{128} \qquad \frac{4}{3} \qquad 2$$

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Arranging these values from lowest to highest gives us the following scale:

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You will notice that there are two different step sizes; the *tone* which is $\frac{9}{8}$ and the *semitone* which is $\frac{256}{243}$. The pattern of this scale is *ttsttts*.

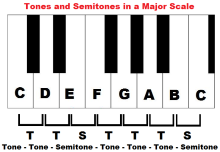

Image source: http://www.piano-keyboard-guide.com/wp-content/uploads/2015/05/tones-and-semitones-in-a-major-scale.png

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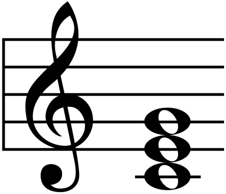

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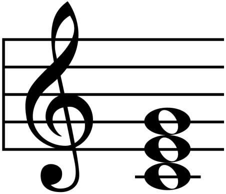

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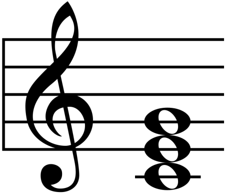

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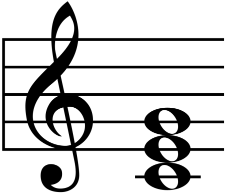

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To make these sound good, we want to replace the awkward fractions $\frac{81}{64}$, $\frac{27}{16}$, and $\frac{243}{128}$. Thus, the 'Classical Just Scale' was created.

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We want the three notes in the chord to be in a clean ratio, which is usually $4:5:6$.

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So $\frac{5}{4}$ is our new third note.

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Now, since the eighth note of the scale is 2, which is an octave above, and therefore twice the first note (1), the ninth note is $\frac{18}{8}$, which is twice the second note ($\frac{9}{8}$).

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So $\frac{15}{8}$ is our new seventh note.

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Thus we end up with this scale:

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These notes lead to triads that are all in the same beautiful ratio! These chords are very pleasing to the ear.

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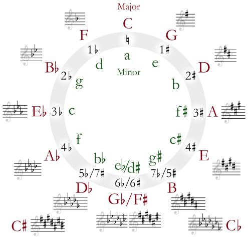

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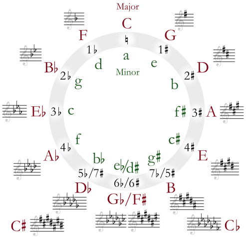

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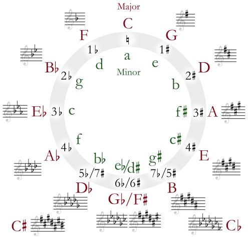
The two scales presented so far do not *transpose* well. That is, they sound strange when they are switched to a different key.



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So, yet another scale was created - the ‘Equal Temperament Scale’.

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Thinking back to the Pythagorean scale, consider a semitone to be 1 step and a tone to be 2 steps. Mathematically we set this up as $t = s^2$.

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Thus, from the pattern *ttsttts* we have 12 steps.

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The fourth note would be $s^4 \times s = s^5$.

Continuing this pattern, the eight and last note, which we know is 2, would be s^{12}.

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Then our scale is as follows:

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Some musicians still like to use the Classical Just scale as they consider it to have a much richer sound, whereas Equal Temperament is a truly mathematical scale!

A Comparison

Today's pianos are tuned around a note called A4, which has a value of 440 Hz.

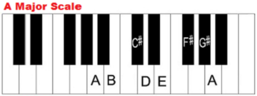

Image source: http://www.piano-keyboard-guide.com

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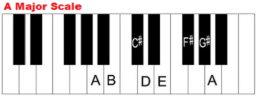

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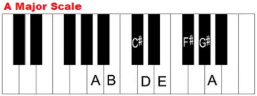

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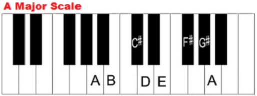

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We can create the three scales we've talked about previously around this note.

Pythagorean A-major Scale

A	B	C#	D	E	F#	G#	A
440 Hz	495 Hz	556.875 Hz	586.667 Hz	660 Hz	742.5 Hz	835.3125 Hz	880 Hz

Classical Just A-major Scale

A	B	C#	D	E	F#	G#	A
440 Hz	495 Hz	550 Hz	586.667 Hz	660 Hz	733.333 Hz	825 Hz	880 Hz

Equal Temperament A-major Scale

A	B	C#	D	E	F#	G#	A
440 Hz	493.883 Hz	554.365 Hz	587.330 Hz	659.255 Hz	739.989 Hz	830.609 Hz	880 Hz

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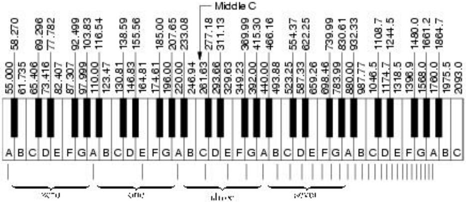
440 Hz	493.883 Hz	554.365 Hz	587.330 Hz	659.255 Hz	739.989 Hz	830.609 Hz	880 Hz

If you were to search "Piano Key Frequencies" on your favourite search engine, you would discover the values listed in the Equal Temperament table!



Thank You!

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