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The fundamental mathematics behind these scales? Fractions!
The sound of a note is based on its frequency.
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This is measured in a unit called the Hertz (Hz). One Hertz means one vibration per second.
An Introduction to Scales

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Thus, the ‘Pythagorean Scale’ was created.
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This leaves us with four missing notes in our eight note scale.

To fill in the last four, we iterate by multiplying by $\frac{3}{2}$, and every time we obtain a value greater than 2, we halve it. We do this because we do not want any notes in our scale higher than an octave above the base. Try it out!
We get the values below:

\[
\begin{array}{cccccccc}
1 & \frac{3}{2} & 9 & \frac{27}{16} & \frac{81}{64} & \frac{243}{128} & \frac{4}{3} & 2 \\
\end{array}
\]

This scale is commonly labeled as: C D E F G A B C

You will notice that there are two different step sizes; the tone which is \( \frac{9}{8} \) and the semitone which is \( \frac{256}{243} \). The pattern of this scale is \( \text{ttsttstt} \).
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The Problem

Some of the *intervals* between notes didn’t sound great, thanks to some of the large or awkward ratios of this scale.

![Musical notes diagram]
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Why are intervals important to us?

Well, a large part of western music is the *chord* or *triad*. The triad is essentially made up of three alternating notes in the scale.

To make these sound good, we want to replace the awkward fractions $\frac{81}{64}$, $\frac{27}{16}$, and $\frac{243}{128}$. Thus, the ‘Classical Just Scale’ was created.
We want the three notes in the chord to be in a clean ratio, which is usually 4 : 5 : 6.
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Let’s examine the chord made up of the first, third, and fifth notes of the scale; replacing the third note, $\frac{81}{64}$, with an unknown:

$$1 : x : \frac{3}{2}$$
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We can make the first and third numbers look like 4 and 6 by multiplying by 4 to get

$$4 : 4x : \frac{12}{2}$$
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Equating the middle term to 5 we find

$$4x = 5$$

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So $\frac{5}{4}$ is our new third note.
Now we examine a chord made up of the fourth, sixth, and eighth notes of the scale, replacing the sixth note, $\frac{27}{16}$, with an unknown:

$$\frac{4}{3} : y : 2$$
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So $\frac{5}{3}$ is our new sixth note.
The only note left to replace is the seventh note, \( \frac{243}{128} \). So we will look at the chord made up of the fifth, seventh, and ninth notes of the scale.
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Now, since the eighth note of the scale is 2, which is an octave above, and therefore twice the first note (1), the ninth note is $\frac{18}{8}$, which is twice the second note ($\frac{9}{8}$).
Classical Just Scale

Replace the seventh note with an unknown:

\[
\frac{3}{2} : z : \frac{18}{8}
\]

We can make the first and third numbers look like 4 and 6 by multiplying by \(\frac{8}{3}\) to get

\[
\frac{24}{3} : z : \frac{144}{24}
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Equating the middle term to 5 we find

\[
z = 15
\]

So \(\frac{15}{8}\) is our new seventh note.
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\frac{24}{6} : \frac{8z}{3} : \frac{144}{24}
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\frac{24}{6} \cdot \frac{8z}{3} \cdot \frac{144}{24}
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\[
\frac{8z}{3} = 5
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So \(\frac{15}{8}\) is our new seventh note.
Thus we end up with this scale:

\[
\begin{align*}
1 & \quad \frac{9}{8} & \quad \frac{5}{4} & \quad \frac{4}{3} & \quad \frac{3}{2} & \quad \frac{5}{3} & \quad \frac{15}{8} & \quad 2
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\]

These notes lead to triads that are all in the same beautiful ratio! These chords are very pleasing to the ear.
Yet, the scale that we are used to today, the one that you would likely have tuned on your piano at home, is NOT the Classical Just.
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The two scales presented so far do not transpose well. That is, they sound strange when they are switched to a different key.
The Problem

Yet, the scale that we are used to today, the one that you would likely have tuned on your piano at home, is NOT the Classical Just.

The two scales presented so far do not transpose well. That is, they sound strange when they are switched to a different key.

So, yet another scale was created - the ‘Equal Temperament Scale’.
Equal Temperament Scale

The idea behind Equal Temperament is to have a consistent step size.
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Thinking back to the Pythagorean scale, consider a semitone to be 1 step and a tone to be 2 steps. Mathematically we set this up as $t = s^2$.

Thus, from the pattern $ttsttts$ we have 12 steps.
Equal Temperament Scale

The first note in our scale is always 1.
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Then, the second note would be $1 \times t = 1 \times s^2 = s^2$. 
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The third note would be $s^2 \times t = s^2 \times s^2 = s^4$.

The fourth note would be $s^4 \times s = s^5$.

Continuing this pattern, the eight and last note, which we know is 2, would be $s^{12}$. 
Equal Temperament Scale

Then we can say:

\[ s^{12} = 2 \]

\[ s = 2^{\frac{1}{12}} \]

Then our scale is as follows:

\[ \begin{array}{ccccccc}
1 & 2 & 1 & 6 & 2 & 1 & 3 \\
7 & 2 & 11 & 12 & 2 & 4 & 10 \\
\end{array} \]

With this scale we get perfect transposition, however you will notice we lose our nice intervals (though they are actually quite close to nice, if you try it out!). Some musicians still like to use the Classical Just scale as they consider it to have a much richer sound, whereas Equal Temperament is a truly mathematical scale!
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We can create the three scales we’ve talked about previously around this note.
## Pythagorean A-major Scale

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C#</th>
<th>D</th>
<th>E</th>
<th>F#</th>
<th>G#</th>
<th>A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hz</td>
<td>440</td>
<td>495</td>
<td>556.875</td>
<td>586.667</td>
<td>660</td>
<td>742.5</td>
<td>835.3125</td>
<td>880</td>
</tr>
</tbody>
</table>

Math in the Real World: Music (9+)
### Classical Just A-major Scale

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If you were to search “Piano Key Frequencies” on your favourite search engine, you would discover the values listed in the Equal Temperament table!
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Visit cemc.uwaterloo.ca for great mathematics resources!