Math in the Real World: Music (7/8)

CEMC

The Connection

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The fundamental mathematics behind these scales? Fractions!

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This is measured in a unit called the Hertz (*Hz*). One Hertz means one vibration per second.

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Essentially, what we're saying is that the higher frequency a note has, the higher pitch or sound it has.

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This ranges from the material the string is made of, to the tension, to the string length, among other influences.

Factors Affecting Frequency: Tension

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So, as the tension of a string increases, the higher the note sounds.

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So, thicker strings have a lower frequency and thus a lower sound than thin strings.

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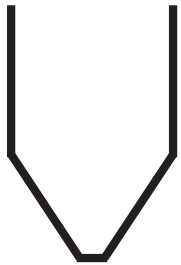
Shorter strings have a higher frequency. The case is similar for wind instruments.

An Experiment

Courtesy of *Scientific American*.

1. Cut one of the two drinking straws so that it is half the length of the other straw.
2. Take one of the straws and flatten about one inch at one end of the straw. You can use your teeth or pinch it between your fingers or fingernails to flatten it.
3. On the same straw, use scissors to make two small, angular cuts, one on each side of the flattened end. This should make the end of the straw be similar to a "V" shape when flattened, but without a pointed tip at the end (the end should have a short, flat, uncut segment left).



4. Repeat this with the other straw so that both have small, angular cuts on one end.

An Experiment

5. Insert the cut end of the longer straw into your mouth. Position the cuts so they're just inside your lips. Then curve your lips down and inward a little and apply light pressure on the straw with your lips.
6. Blow through the straw. You may need to move the straw around slightly to locate the best position for creating your musical note. It might take some practice and repeated tries to produce a constant, single note.
7. Now blow through the shorter straw using the same method. Again, you might need to try blowing through the straw a few times to make it produce a constant, single note.

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The Segue

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Thus, the 'Pythagorean Scale' was created.

Building the Pythagorean Scale

1. Start with a base frequency of value 1.
2. The top frequency should be double. So give it a value of 2.
3. Multiply the base frequency by $\frac{3}{2}$ and divide the top frequency by $\frac{3}{2}$.
4. To fill in the last four notes, iterate by multiplying by $\frac{3}{2}$, and every time you obtain a value *greater than* 2, halve it. We do this because we do not want any notes in our scale higher than an octave above the base.

Try it out!

Pythagorean Scale

We get the values below:

$$1 \quad \frac{3}{2} \quad \frac{9}{8} \quad \frac{27}{16} \quad \frac{81}{64} \quad \frac{243}{128} \quad \frac{4}{3} \quad 2$$

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This scale is commonly labeled as:

C D E F G A B C

Pythagorean Scale

However, we can start on any of these notes, as long as we follow the pattern. For example, we could have labelled the scale

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Now, with this model, what would be the frequency of the notes in the scale?

Pythagorean Scales on the Piano

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Without going too in-depth into music theory, we are going to create what is call the A-major scale, which contains F, C, and G sharps (denoted by the # symbol).

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This is what we end up with:

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Try it out:

1. Create the D-major scale. Set the base note to frequency 73.4162 Hz, and note the scale has an F# and C#.
2. Create the F-major scale. Set the base note to frequency 87.3071 Hz, and note the scale has a B♭ (B flat).

Thank You!

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