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Music and Mathematics (Grade 7/8)

Many of you probably play instruments! But did you know that the foundations of music are built with mathematics? One of the first things you learn when you start with instruments is something called a *scale*. Boiled down to basics, we can say a scale is a sequence of notes played over an *octave*. The scales that we play have evolved over time and vary by region. The fundamental mathematics behind these scales? Fractions!

Frequency

The sound of a note is based on its *frequency*. In terms of a string, like the ones plucked on a guitar or struck by a hammer in a piano, this means the number of vibrations up and down per second. This is measured in a unit called the Hertz (*Hz*). One Hertz means one vibration per second. A scale starts with a base frequency and ends on a note that is twice the frequency. We call the difference between a note and another note that is twice the frequency an octave. Essentially, what we're saying is that the higher frequency a note has, the higher pitch or sound it has.

The frequency of a note (which is the frequency of the string) depends on a number of factors. This ranges from the material the string is made of, to the tension, to the string length, among other influences.

What effect does the tension on the string have? Imagine you're holding the ends of an elastic band, and you're holding it so it sags. If your friend plucks the elastic, it barely vibrates, if at all. As you tighten the band by pulling the ends farther apart, the elastic will vibrate faster and faster when it's plucked, and you'll hear that the sound it makes gets higher and higher! So, as the tension of a string increases, the higher the note sounds.

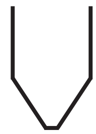
What about string thickness? Imagine you have two skipping ropes, one that is very thin and light, and one that is quite thick and heavy. Which one would you be able to turn faster? The thin, light one of course! The thick, heavy one is much harder to turn around (it takes more effort). In this way, thicker strings vibrate more slowly than thin ones. So, thicker strings have a lower frequency and thus a lower sound than thin strings.

Lastly, how does the length of a string affect its sound? When playing a guitar, the guitarist places their fingers higher and lower along the guitar to produce different notes. Similarly, in a grand piano, the length of strings in the piano changes as you go up the keyboard. What you will observe if you look inside a grand piano, or watch a guitarist's fingers, is that the shorter the string, the higher the note produced. Shorter strings have a higher frequency. The case is similar for wind instruments.

We can try the following experiment using two straws, courtesy of *Scientific American*.

- Cut one of the two drinking straws so that it is half the length of the other straw.
- Take one of the straws and flatten about one inch at one end of the straw. You can use your teeth or pinch it between your fingers or fingernails to flatten it.
- On the same straw, use scissors to make two small, angular cuts, one on each side of the flattened

end. This should make the end of the straw be similar to a "V" shape when flattened, but without a pointed tip at the end (the end should have a short, flat, uncut segment left).



- Repeat this with the other straw so that both have small, angular cuts on one end.
- Insert the cut end of the longer straw into your mouth. Position the cuts so they're just inside your lips. Then curve your lips down and inward a little and apply light pressure on the straw with your lips.
- Blow through the straw. You may need to move the straw around slightly to locate the best position for creating your musical note. It might take some practice and repeated tries to produce a constant, single note.
- Now blow through the shorter straw using the same method. Again, you might need to try blowing through the straw a few times to make it produce a constant, single note.

You should notice that the shorter straw produces a note that is higher than the one produced by the longer straw. In fact, since the short straw is exactly half the length of the long straw, you are producing a note that is twice the frequency - an octave higher!

But what exactly are the notes in a scale? Well, it depends on what type of scale you're talking about. This is where fractions really come into play. One classic scale was created by the ancient Greeks. They noticed that strings in the ratio $\frac{3}{2}$ sounded quite good together. Thus, the 'Pythagorean Scale' was created.

Pythagorean Scale

To create this scale, start with a base frequency. For simplicity, give this a value of 1. As we know, an octave ends at double the frequency, which in this case is 2. As the ratio $\frac{3}{2}$ is nice, we multiply the base frequency by $\frac{3}{2}$ and divide the top frequency by $\frac{3}{2}$. This leaves us with four missing notes in our eight note scale. To fill in the last four, we iterate by multiplying by $\frac{3}{2}$, and every time we obtain a value *greater than* 2, we halve it. We do this because we do not want any notes in our scale higher than an octave above the base. **Try it out!** We get the values below:

$$1 \quad \frac{3}{2} \quad \frac{9}{8} \quad \frac{27}{16} \quad \frac{81}{64} \quad \frac{243}{128} \quad \frac{4}{3} \quad 2$$

Arranging these values from lowest to highest gives us the following scale:

$$1 \quad \frac{9}{8} \quad \frac{81}{64} \quad \frac{4}{3} \quad \frac{3}{2} \quad \frac{27}{16} \quad \frac{243}{128} \quad 2$$

You will notice that there are two different step sizes; the *tone* which is $\frac{9}{8}$ and the *semitone* which is $\frac{256}{243}$. The pattern of this scale is *ttsttts*.

This scale is commonly labeled as:

C D E F G A B C

However, we can start on any of these notes, as long as we follow the pattern. For example, we could have labelled the scale

A B C D E F G A

Now, with this model, what would be the frequency of the notes in the scale? Today's pianos are tuned around a note called A4, which has a value of 440 Hz. Imagine the piano were to be tuned on a Pythagorean scale. (In actuality, most pianos today are tuned to the *Equal Temperament Scale.*) Without going too in-depth into music theory, we are going to create what is call the A-major scale, which contains F, C, and G sharps (denoted by the # symbol). To do this, we set our base note (which we normal give a value of 1) to 440 Hz. Then, we multiply each of the scale values above by 440 Hz. We are essentially scaling our scale. This is what we end up with:

A	B	C#	D	E	F#	G#	A
440 Hz	495 Hz	556.875 Hz	586.667 Hz	660 Hz	742.5 Hz	835.3125 Hz	880 Hz

Try it out:

1. Create the D-major scale. Set the base note to frequency 73.4162 Hz, and note the scale has an F# and C#.

2. Create the F-major scale. Set the base note to frequency 87.3071 Hz, and note the scale has a B♭ (B flat).