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Music and Mathematics (Grade 9+)

Many of you probably play instruments! But did you know that the foundations of music are built with mathematics? One of the first things you learn when you start with instruments is something called a 'scale'. The scales that we play have evolved over time and vary by region. The fundamental mathematics behind these scales? Fractions!

The sound of a note is based on its *frequency*. In terms of a string, like the ones plucked on a guitar or struck by a hammer in a piano, this means the number of vibrations per second (*Hz*). A scale starts with a base frequency and ends on a note that is twice the frequency (the scale runs through an *octave*).

We will focus on three types of Western scales, as they evolved through time. Our story begins with the ancient Greeks. They noticed that strings in the ratio $\frac{3}{2}$ sounded quite good together. Thus, the 'Pythagorean Scale' was created.

Pythagorean Scale

To create this scale, start with a base frequency. For simplicity, give this a value of 1. As we know, an octave ends at double the frequency, which in this case is 2. As the ratio $\frac{3}{2}$ is nice, we multiply the base frequency by $\frac{3}{2}$ and divide the top frequency by $\frac{3}{2}$. This leaves us with four missing notes in our eight note scale. To fill in the last four, we iterate by multiplying by $\frac{3}{2}$, and every time we obtain a value *greater than* 2, we halve it. We do this because we do not want any notes in our scale higher than an octave above the base. **Try it out!** We get the values below:

$$1 \quad \frac{3}{2} \quad \frac{9}{8} \quad \frac{27}{16} \quad \frac{81}{64} \quad \frac{243}{128} \quad \frac{4}{3} \quad 2$$

Arranging these values from lowest to highest gives us the following scale:

$$1 \quad \frac{9}{8} \quad \frac{81}{64} \quad \frac{4}{3} \quad \frac{3}{2} \quad \frac{27}{16} \quad \frac{243}{128} \quad 2$$

This scale is commonly labeled as:

C D E F G A B C

You will notice that there are two different step sizes; the *tone* which is $\frac{9}{8}$ and the *semitone* which is $\frac{256}{243}$. The pattern of this scale is *ttsttts*.

However, there was a problem with this type of scale. Some of the *intervals* between notes didn't sound great, thanks to some of the large or awkward ratios of this scale. Why are intervals important to us? Well, a large part of western music is the *chord* or *triad*. The triad is essentially made up of three alternating notes in the scale. To make these sound good, we want to replace the awkward fractions $\frac{81}{64}$, $\frac{27}{16}$, and $\frac{243}{128}$. Thus, the 'Classical Just Scale' was created.

Classical Just Scale

As mentioned before, an important part of music is the chord/triad. We want the three notes in the chord to be in a clean ratio, which is usually $4:5:6$. Lets examine the chord made up of the first, third, and fifth notes of the scale; replacing the third note, $\frac{81}{64}$, with an unknown.

$$1 : x : \tfrac{3}{2}$$

We can make the first and third numbers look like 4 and 6 by multiplying by 4 to get

$$4 : 4x : \tfrac{12}{2}$$

Equating the middle term to 5 we find

$$4x = 5$$
$$x = \frac{5}{4}$$

So $\frac{5}{4}$ is our new third note.

Now we examine a chord made up of the fourth, sixth, and eighth notes of the scale, replacing the sixth note, $\frac{27}{16}$ with an unknown.

$$\tfrac{4}{3} : y : 2$$

We can make the first and third numbers look like 4 and 6 by multiplying by 3 to get

$$\tfrac{12}{3} : 3y : 6$$

Equating the middle term to 5 we find

$$3y = 5$$
$$y = \frac{5}{3}$$

So $\frac{5}{3}$ is our new sixth note.

The only note left to replace is the seventh note, $\frac{243}{128}$. So we will look at the chord made up of the fifth, seventh, and ninth notes of the scale. Now, since the eighth note of the scale is 2, which is an octave above, and therefore twice the first note (1), the ninth note is $\frac{18}{8}$, which is twice the second note ($\frac{9}{8}$). Replace the seventh note with an unknown.

$$\tfrac{3}{2} : z : \tfrac{18}{8}$$

We can make the first and third numbers look like 4 and 6 by multiplying by $\frac{8}{3}$ to get

$$\tfrac{24}{6} : \tfrac{8z}{3} : \tfrac{144}{24}$$

Equating the middle term to 5 we find

$$\frac{8z}{3} = 5$$
$$z = \frac{15}{8}$$

So $\frac{15}{8}$ is our new seventh note.

Thus we end up with this scale:

$$1 \quad \tfrac{9}{8} \quad \tfrac{5}{4} \quad \tfrac{4}{3} \quad \tfrac{3}{2} \quad \tfrac{5}{3} \quad \tfrac{15}{8} \quad 2$$

These notes lead to triads that are all in the same beautiful ratio! These chords are very pleasing to the ear.

Yet, the scale that we are used to today, the one that you would likely have tuned on your piano at home, is NOT the Classical Just. The two scales presented so far do not *transpose* well. That is, they sound strange when they are switched to a different key. So, yet another scale was created - the 'Equal Temperament Scale'.

Equal Temperament Scale

The idea behind Equal Temperament is to have a consistent step size. Thinking back to the Pythagorean scale, consider a semitone to be 1 step and a tone to be 2 steps. Mathematically we set this up as $t = s^2$. Thus, from the pattern *ttsttts* we have 12 steps. The first note in our scale is always 1. Then, the second note would be $1 \times t = 1 \times s^2 = s^2$. The third note would be $s^2 \times t = s^2 \times s^2 = s^4$. The fourth note would be $s^4 \times s = s^5$. Continuing this pattern, the eight and last note, which we know is 2, would be s^{12}. Then we can say

$$s^{12} = 2$$
$$s = 2^{\frac{1}{12}}$$

Then our scale is as follows:

$$1 \quad 2^{\frac{1}{6}} \quad 2^{\frac{1}{3}} \quad 2^{\frac{5}{12}} \quad 2^{\frac{7}{12}} \quad 2^{\frac{3}{4}} \quad 2^{\frac{11}{12}} \quad 2$$

With this scale we get perfect transposition, however you will notice we lose our nice intervals (though they are actually quite close to nice, if you try it out!). Some musicians still like to use the Classical Just scale as they consider it to have a much richer sound, whereas Equal Temperament is a truly mathematical scale!

A Comparison

Today's pianos are tuned around a note called A4, which has a value of 440 Hz. Without going too in-depth into music theory, we are going to create what is call the A-major scale. A-major contains F, C, and G sharps (denoted by the # symbol). We can create the three scales we've talked about above around this note. Check them out below:

Pythagorean Scale

A	B	C#	D	E	F#	G#	A
440 Hz	495 Hz	556.875 Hz	586.667 Hz	660 Hz	742.5 Hz	835.3125 Hz	880 Hz

Classical Just Scale

A	B	C#	D	E	F#	G#	A
440 Hz	495 Hz	550 Hz	586.667 Hz	660 Hz	733.333 Hz	825 Hz	880 Hz

Equal Temperament Scale

A	B	C#	D	E	F#	G#	A
440 Hz	493.883 Hz	554.365 Hz	587.330 Hz	659.255 Hz	739.989 Hz	830.609 Hz	880 Hz

If you were to search "Piano Key Frequencies" on your favourite search engine, you would discover the values listed in the Equal Temperament table!