Math in the Real World: Digital Imaging

CEMC
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The Connection

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We will start by discussing how these images are stored on a computer and then we will move on to how this data is transformed for different purposes.
The Binary Number System

The number system that we use on a daily basis is known as base 10 or decimal. However, computers use the binary or base 2 system.

Let's take for example the number 473. Here's how we would break this number down in both systems:

**Decimal**

\[ 473 = 4(10^2) + 7(10^1) + 3(10^0) \]
\[ 473 = (473)_{10} \]

**Binary**

\[ 473 = 1(2^8) + 1(2^7) + 1(2^6) + 0(2^5) + 1(2^4) + 1(2^3) + 0(2^2) + 0(2^1) + 1(2^0) \]
\[ 473 = (111011001)_2 \]

As you can see, binary gives us a string of 1's and 0's. Try converting 255 to binary!
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As you can see, binary gives us a string of 1’s and 0’s. **Try converting 255 to binary!** Make sure you start by finding the largest power of 2 that is less than 255.
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The Colour Scales

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The simplest colour scale is *greyscale*. Technically, this is a shade scale that ranges from pure black to pure white. Black is 0 and white is 255.
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Image source: http://www.whydomath.org/node/wavlets/images/grayrange.gif
When we rewatch our favourite *Harry Potter* movie, we watch it in full colour!
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Image source: https://www.medialooks.com/mformats/docs/images/CK_color_cube.png
The Colour Scales

For example, the University of Waterloo’s official gold colour is (255, 213, 79):

[Image of yellow square]
The Colour Scales

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Try it out:

1. What is the binary coordinate representation of Waterloo Gold?
2. What would black (on the RGB scale) be in decimal?
3. What would black (on the RGB scale) be in binary?
All digital images are made up of tiny pixels. Your screen is lit up with these tiny squares, each with different colour values.
Storing Information: Pixels and Bits

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The more pixels per inch (PPI) there are (for the same image size in different resolutions), the higher quality your image is. Your image looks less grainy, as you are able to encode more detailed colours with your smaller pixels in the same area.
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Take a look at two examples of PPI below:

![Image source: http://s01.shiftdelete.net/img/content/16-10/04/ppi_karsilastirmasi.jpg](http://s01.shiftdelete.net/img/content/16-10/04/ppi_karsilastirmasi.jpg)
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Eight bits make up one byte, and $2^{10} = 1024$ bytes make up one kilobyte (KB).

Furthermore, $(2^{10})^2 = 1024^2 = 1048576$ bytes make up one megabyte (MB). The megabyte, along with the gigabyte (GB) are probably the most familiar storage units.
Storing Information: Pixels and Bits

Try it out:
1. How many bits are in one black and white (greyscale) pixel?
2. How many bits are in one colour (RGB) pixel?
3. How many bits are in the HD colour image pictured above?
4. How many bits are in the 4K colour image pictured above?
5. What is the file size of the HD image in MB?
6. What is the file size of the 4K image in MB?
7. What percentage of the size of the HD file is the size of the 4K file?
8. How many bytes do you think there are in 1 GB?
When we upload our pictures onto social media, they get compressed so they take up less space on the social media company’s servers.
A Component of Compression: Huffman Coding

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One simplistic method of data compression that is occasionally utilized: Huffman Coding.
A Component of Compression: Huffman Coding

This method can be computerized to deal with large amounts of data, but for workability, we will go back to greyscale images.

---

Examine the section of pixels captured below, along with its numeric greyscale values:

```
51 155 132 86
132 51 196 51
216 86 132 51
```

To start our Huffman Coding, we need to create a frequency table:

<table>
<thead>
<tr>
<th>Value</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>51</td>
<td>4</td>
</tr>
<tr>
<td>132</td>
<td>3</td>
</tr>
<tr>
<td>86</td>
<td>2</td>
</tr>
<tr>
<td>155</td>
<td>1</td>
</tr>
<tr>
<td>196</td>
<td>1</td>
</tr>
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<td>1</td>
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```
Relative Frequency
```

```
/12 = 0.33
/12 = 0.25
/12 = 0.17
/12 = 0.08
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Next, we sort the values by relative frequency (smallest to largest):
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```
216 (0.08)  196 (0.08)  155 (0.08)  86 (0.17)  132 (0.25)  51 (0.33)
```

Now, add the relative frequencies of the two leftmost nodes and create a new node with two children:
A Component of Compression: Huffman Coding

Repeat the previous step:
A Component of Compression: Huffman Coding

Keep repeating until there is a single node of frequency 1 at the top:
Lastly, label the branches connecting nodes. Left branches get a 0, right branches get a 1.
Originally, the value of 132 was 10000100. From our Huffman Coding, the new value of 132 is 01 (see the path highlighted in red below). We have saved 6 bits for every occurrence of 132.
A Component of Compression: Huffman Coding

Try it out:

1. How many bits were originally used to store the section of greyscale pixels?
2. Create a table listing the original pixel values (in decimal) and their new Huffman values.
3. How many bits are used to store the Huffman coded section of pixels?
4. What percentage of the size of the original file is the size of the Huffman file?
5. What made Huffman Coding so effective with this specific section of pixels? In what situation would Huffman Coding not be as effective?
6. Create the Huffman tree for the frequency table below and list the new Huffman values.

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<thead>
<tr>
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</thead>
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<tr>
<td>115</td>
<td>5</td>
<td>5/10 = 0.5</td>
</tr>
<tr>
<td>97</td>
<td>3</td>
<td>3/10 = 0.3</td>
</tr>
<tr>
<td>102</td>
<td>1</td>
<td>1/10 = 0.1</td>
</tr>
<tr>
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Once again, let’s examine a section of pixels:
Now, we’re going to label the pixels with coordinates as follows:

\[ f(x, y) \]

We can use function notation to represent our grid of pixels. The function \( f(x, y) \) calls on the pixel located at \((x, y)\) and gives us the RGB colour code for that pixel. For example, \( f(0, 1) = (0, 0, 0) \) which we know is black. We can use this notation to manipulate the pixels.
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![function notation diagram]
Now, we’re going to label the pixels with coordinates as follows:

\[
\begin{array}{cccc}
1 & 1 & 0 & 0 \\
2 & 2 & 3 & \\
0 & 0 & 0 & 0 \\
\end{array}
\]

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For example, \( f(0, 1) = (0, 0, 0) \) which we know is black. We can use this notation to manipulate the pixels.
Let’s flip/mirror the section of pixels horizontally. We will call the function that represents our new section of pixels $h(x, y)$. 

From these sample points, we can see that the horizontally flipped image is 

$$h(x, y) = f(3 - x, y).$$
Let’s flip/mirror the section of pixels horizontally. We will call the function that represents our new section of pixels $h(x, y)$.

Choosing a few points as examples, we see that $h(0, 0) = f(3, 0)$, $h(1, 2) = f(2, 2)$, and $h(3, 2) = f(0, 2)$. 
Transforming Images with Functions

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From these sample points, we can see that the horizontally flipped image is $h(x, y) = f(3 - x, y)$. 

![Image](image_url)
Try it out:

1. What function \( v(x, y) \) flips/mirrors the original section of pixels \( f(x, y) \) vertically?

2. What function \( r(x, y) \) rotates \( f(x, y) \) \( 180^\circ \)? (Hint: Don't think of it as a rotation.)

3. The \( 11 \times 9 \) grid of pixels \( a(x, y) \) is shown to the left. Draw \( b(x, y) = a(2x, 2y) \) with the restrictions \( x \leq 4, y \leq 5 \). (Hint 1: \( b(x, y) \) should be a \( 6 \times 5 \) grid. Hint 2: You can immediately eliminate the 69 squares that don't show up in \( b(x, y) \) if you think about what is happening with the function.)
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