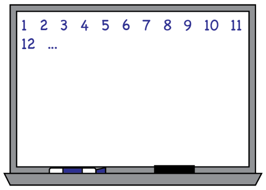

Problem of the Week

Problem D and Solution

Wipe Away 2

Problem

Ajay writes the positive integers from 1 to 1000 on a whiteboard. Jamilah then erases all the numbers that are multiples of 9. Magdalena then erases all the remaining numbers that contain the digit 9. How many numbers are left on the whiteboard?

NOTE: In solving this problem, it may be helpful to use the fact that a number is divisible by 9 exactly when the sum of its digits is divisible by 9. For example, the number 214 578 is divisible by 9 since $2 + 1 + 4 + 5 + 7 + 8 = 27$, which is divisible by 9. In fact, $214\,578 = 9 \times 23\,842$.

Solution

We first calculate the number of integers that Jamilah erases, which is the number of multiples of 9 between 1 and 1000. Since $1000 = (111 \times 9) + 1$, there are 111 multiples of 9 between 1 and 1000. Thus, Jamilah erases 111 numbers from the whiteboard.

Now let's figure out how many of the integers from 1 to 1000 contain the digit 9. The integers from 1 to 100 that contain the digit 9 are $9, 19, \ldots, 79, 89$ as well as $90, 91, \ldots, 97, 98, 99$. Thus, there are 19 positive integers from 1 to 100 that contain the digit 9. Since there are 19 integers from 1 to 100 that contain the digit 9, it follows that there are $19 \times 9 = 171$ integers from 1 to 899 that contain the digit 9.

Between 900 and 1000, there are 100 integers that contain the digit 9, namely, every number except for 1000. Thus, in total, $171 + 100 = 271$ of the integers from 1 to 1000 contain the digit 9.

However, some of the integers that contain the digit 9 are also multiples of 9, so were erased by Jamilah. To determine how many of these such numbers there are, we use the fact that a number is divisible by 9 exactly when the sum of its digits is divisible by 9.

- The only one-digit number that contains the digit 9 and is also a multiple of 9 is 9 itself.
- The only two-digit numbers that contain the digit 9 and are also multiples of 9 are 90 and 99.
- To find the three-digit numbers that contain the digit 9 and are also multiples of 9, we will look at their digit sum.

- **Case 1:** Three digit-numbers with a digit sum of 9:
  The only possibility is 900. Thus, there is 1 number.
- **Case 2:** Three digit-numbers with a digit sum of 18:
  * If two of the digits are 9, then the other digit must be 0. The only possibilities are 909 and 990. Thus, there are 2 numbers.
  * If only one of the digits is 9, then the other two digits must add to 9. The possible digits are $9,\ 4,\ 5$, or $9,\ 3,\ 6$, or $9,\ 2,\ 7$, or $9,\ 8,\ 1$. For each of these sets of digits, there are 3 choices for the hundreds digit. Once the hundreds digit is chosen, there are 2 choices for the tens digit, and then the remaining digit must be the ones digit. Thus, there are $3 \times 2 = 6$ possible three-digit numbers for each set of digits. Since there are 4 sets of digits, then there are $4 \times 6 = 24$ possible numbers.
- **Case 3:** Three digit-numbers with a digit sum of 27:
  The only possibility is 999. Thus, there is 1 number.

Therefore, there are $1 + 2 + 24 + 1 = 28$ three-digit numbers from 1 to 1000 that contain the digit 9, and are also multiples of 9.

Thus, there are $1 + 2 + 28 = 31$ numbers that contain the digit 9, but were erased by Jamilah. It follows that Magdalena erases $271 - 31 = 240$ numbers from the whiteboard.

Hence, the number of numbers left on the whiteboard is $1000 - 111 - 240 = 649$.