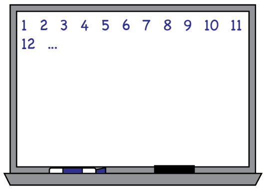

Problem of the Week

Problem E and Solution

Wipe Away 3

Problem

Tyra writes consecutive positive integers on a whiteboard starting with the integer 1. However, when she writes a number that is a multiple of 9, or contains the digit 9, Juliana immediately erases it. If they continue this for a long time, what is the 400^{th} number that Juliana will erase?

NOTE: In solving this problem, it may be helpful to use the fact that a number is divisible by 9 exactly when the sum of its digits is divisible by 9. For example, the number 214 578 is divisible by 9 since $2+1+4+5+7+8=27$, which is divisible by 9. In fact, $214\,578 = 9 \times 23\,842$.

Solution

We first consider the integers between 1 and 999, inclusive. Since $999 = 111 \times 9$, there are 111 multiples of 9 between 1 and 999.

Now let's figure out how many of the integers from 1 to 999 contain the digit 9. The integers from 1 to 99 that contain the digit 9 are $9,\ 19, \ldots, 79,\ 89$ as well as $90,\ 91, \ldots, 97\ ,98,\ 99$. Thus, there are 19 positive integers from 1 to 99 that contain the digit 9. Since there are 19 integers from 1 to 99 that contain the digit 9, it follows that there are $19 \times 9 = 171$ integers from 1 to 899 that contain the digit 9.

Between 900 and 999, every integer contains the digit 9. Thus, there are 100 numbers that contain the digit 9. Thus, in total, $171 + 100 = 271$ of the integers from 1 to 999 contain the digit 9.

However, some of the integers that contain the digit 9 are also multiples of 9, so were counted twice. To determine how many of these such numbers there are, we use the fact that a number is divisible by 9 exactly when the sum of its digits is divisible by 9.

- The only one-digit number that contains the digit 9 and is also a multiple of 9 is 9 itself.
- The only two-digit numbers that contain the digit 9 and are also multiples of 9 are 90 and 99.
- To find the three-digit numbers that contain the digit 9 and are also multiples of 9, we will look at their digit sum.
  - **Case 1:** Three digit-numbers with a digit sum of 9:
    The only possibility is 900. Thus, there is 1 number.
  - **Case 2:** Three digit-numbers with a digit sum of 18:
    * If two of the digits are 9, then the other digit must be 0. The only possibilities are 909 and 990. Thus, there are 2 numbers.
    * If only one of the digits is 9, then the other two digits must add to 9. The possible digits are $9,\ 4,\ 5$, or $9,\ 3,\ 6$, or $9,\ 2,\ 7$, or $9,\ 8,\ 1$. For each of these sets of digits, there are 3 choices for the hundreds digit. Once the hundreds

digit is chosen, there are 2 choices for the tens digit, and then the remaining digit must be the ones digit. Thus, there are $3 \times 2 = 6$ possible three-digit numbers for each set of digits. Since there are 4 sets of digits, then there are $4 \times 6 = 24$ possible numbers.

– **Case 3:** Three digit-numbers with a digit sum of 27:
  The only possibility is 999. Thus, there is 1 number.

Therefore, there are $1 + 2 + 24 + 1 = 28$ three-digit numbers from 1 to 999 that contain the digit 9, and are also multiples of 9.

Thus, there are $1 + 2 + 28 = 31$ integers from 1 to 999 that contain the digit 9, and are also multiples of 9. It follows that Juliana erases $111 + 271 - 31 = 351$ of the numbers from 1 to 999 from the whiteboard. Since we are looking for the 400^{th} number that Juliana erases, we need to keep going.

Next, we consider the integers between 1000 and 1099, inclusive. Since $1099 = (122 \times 9) + 1$, there are 122 multiples of 9 between 1 and 1099. Since there are 111 multiples of 9 between 1 and 999, it follows that there are $122 - 111 = 11$ multiples of 9 between 1000 and 1099. The integers between 1000 and 1099 that contain the digit 9 are $1009,\ 1019, \ldots, 1079,\ 1089$ as well as $1090,\ 1091, \ldots, 1097\ , 1098,\ 1099$. Thus, there are 19 integers from 1000 to 1099 that contain the digit 9. Of these, the only integers that are also multiples of 9 are 1089 and 1098. Thus, Juliana erases $11 + 19 - 2 = 28$ of the numbers from 1000 to 1099 from the whiteboard. In total, she has now erased $351 + 28 = 379$ numbers.

Next, we consider the integers between 1100 and 1189, inclusive. Since $1189 = (132 \times 9) + 1$, there are 132 multiples of 9 between 1 and 1189. Since there are 122 multiples of 9 between 1 and 1099, it follows that there are $132 - 122 = 10$ multiples of 9 between 1100 and 1189. The integers between 1100 and 1189 that contain the digit 9 are $1109,\ 1119, \ldots, 1179,\ 1189$. Thus there are 9 integers from 1100 to 1189 that contain the digit 9. The only one of these that is also a multiple of 9 is 1179. Thus, Juliana erases $10 + 9 - 1 = 18$ of the numbers from 1100 to 1189 from the whiteboard. In total, she has now erased $379 + 18 = 397$ numbers.

The next three numbers that Juliana will erase are $1190,\ 1191$, and 1192. Thus, the 400^{th} number that Juliana erases is 1192.