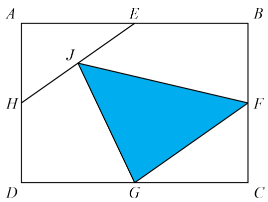


Problem of the Week
Problem E and Solution
Stained Glass

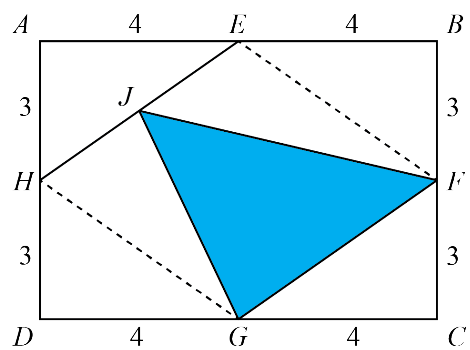
Problem

A stained glass window hanging is in the shape of a rectangle with a length of 8 cm and a width of 6 cm.

Rectangle $ABCD$ represents the window hanging with $AB = 8$ and $BC = 6$. The points E, F, G, and H are the midpoints of sides AB, BC, CD, and AD, respectively. The point J is the midpoint of line segment EH. Triangle FGJ is coloured blue. Determine the area of the blue triangle.

Solution

Solution 1



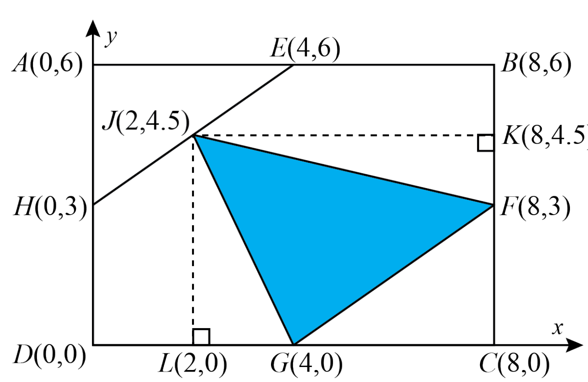
Since $ABCD$ is a rectangle and $AB = 8$, it follows that $AE = EB = DG = GC = 4$. Similarly, since $BC = 6$, it follows that $BF = FC = AH = HD = 3$.
Consider the four corner triangles, $\triangle HAE$, $\triangle EBF$, $\triangle FCG$, and $\triangle GDH$. Each of these triangles is a right-angled triangle with base 4 and height 3. Therefore, the total area of these four triangles is equal to $4 \times \frac{4\times 3}{2} = 24$.

The length of the hypotenuse of each of the four corner triangles is equal to $\sqrt{3^2 + 4^2} = 5$. Thus, $EF = FG = GH = EH = 5$, so $EFGH$ is a rhombus. Thus $EH \parallel FG$. The area of rhombus $EFGH$ is equal to the area of rectangle $ABCD$ minus the area of the four corner triangles. Thus, the area of rhombus $EFGH$ is $8 \times 6 - 24 = 24$.

Let h be the perpendicular distance between FG and EH. Then the area of rhombus $EFGH$ is $h \times FG$. Thus, $h \times 5 = 24$.

Triangle FGJ has base FG and height h, so its area is equal to $\frac{h\times FG}{2} = \frac{h\times 5}{2} = \frac{24}{2} = 12 \text{ cm}^2$.

Solution 2



In this solution we will use analytic geometry and set the coordinates of D to $(0,0)$. Then $A(0,6)$, $B(8,6)$, and $C(8,0)$ are the other corners of the rectangle. The midpoints E, F, G, and H have coordinates $(4,6)$, $(8,3)$, $(4,0)$, and $(0,3)$, respectively. Then J has coordinates $(2,4.5)$. Let K have coordinates $(8,4.5)$, and L have coordinates $(2,0)$. Then $JKCL$ is a rectangle.

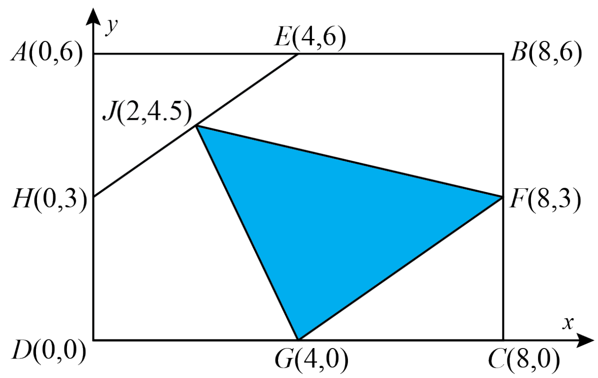
We can then calculate the area of $\triangle FGJ$ as follows.

$$\begin{aligned}\text{Area } \triangle FGJ &= \text{Area } JKCL - \text{Area } \triangle JKF - \text{Area } \triangle FCG - \text{Area } \triangle GLJ \\ &= JK \times CK - \frac{JK \times KF}{2} - \frac{FC \times CG}{2} - \frac{GL \times LJ}{2} \\ &= 6 \times 4.5 - \frac{6 \times 1.5}{2} - \frac{3 \times 4}{2} - \frac{2 \times 4.5}{2} \\ &= 27 - 4.5 - 6 - 4.5 \\ &= 12\end{aligned}$$

Therefore, the area of $\triangle FGJ$ is equal to 12 cm^2.

Solution 3

This solution also uses analytic geometry. As in Solution 2, set the coordinates of D to $(0,0)$. Then $A(0,6)$, $B(8,6)$, and $C(8,0)$ are the other corners of the rectangle. The midpoints E, F, G, and H have coordinates $(4,6)$, $(8,3)$, $(4,0)$, and $(0,3)$, respectively. Then J has coordinates $(2,4.5)$.



The base of $\triangle FGJ$ is equal to the length of FG. Since $\triangle FCG$ is a right-angled triangle, $FG = \sqrt{CF^2 + CG^2} = \sqrt{3^2 + 4^2} = 5$. Line segments EH and FG each have a slope of $\frac{3}{4}$, so it follows that they are parallel. Thus, the height of $\triangle FGJ$ is equal to the perpendicular distance between EH and FG.

The line passing through F and G has slope $\frac{3}{4}$. The line perpendicular to FG, passing through G has slope $-\frac{4}{3}$ and y-intercept $\frac{16}{3}$. Therefore, its equation is $y = -\frac{4}{3}x + \frac{16}{3}$.

The line passing through EH has slope $\frac{3}{4}$ and y-intercept 3. Therefore, its equation is $y = \frac{3}{4}x + 3$. We can then determine the point of intersection of $y = \frac{3}{4}x + 3$ and $y = -\frac{4}{3}x + \frac{16}{3}$ by setting $\frac{3}{4}x + 3 = -\frac{4}{3}x + \frac{16}{3}$.

We multiply both sides of this equation by 12 and solve for x:

$$\begin{aligned}9x + 36 &= -16x + 64 \\ 25x &= 28 \\ x &= \frac{28}{25}\end{aligned}$$

The y-coordinate for this intersection point is then $y = \frac{3}{4}\left(\frac{28}{25}\right) + 3 = \frac{21}{25} + 3 = \frac{96}{25}$.

Then, the height of $\triangle FGJ$ is equal to the distance between $\left(\frac{28}{25}, \frac{96}{25}\right)$ and $G(4,0)$, which is

$$\sqrt{\left(\frac{96}{25} - 0\right)^2 + \left(\frac{28}{25} - 4\right)^2} = \sqrt{\frac{9216}{625} + \frac{5184}{625}} = \sqrt{\frac{14\,400}{625}} = \sqrt{\frac{576}{25}} = \frac{24}{5}$$

Therefore, the area of $\triangle FGJ$ is equal to $\frac{1}{2} \times 5 \times \frac{24}{5} = 12$ cm^2.

Extension: Suppose $AB = p$ and $BC = q$, for some real numbers p and q. Determine the area of $\triangle FGJ$.