# Problem of the Week
## Problem E and Solution
## Coin Combinations

**Problem**

In Canada, a \$2 coin is called a toonie, a \$1 coin is called a loonie, and a 25¢ coin is called a quarter. Four quarters have a value of \$1.

How many different combinations of toonies, loonies, and/or quarters have a total value of \$100?

## Solution

We will break the solution into cases based on the number of \$2 coins used. For each case, we will count the number of possibilities for the number of \$1 and 25¢ coins.

The maximum number of \$2 coins we can use is 50, since $\$2 \times 50 = \$100$. If we use 50 \$2 coins, then we do not need any \$1 or 25¢ coins. Therefore, there is only one way to make a total of \$100 if there are 50 \$2 coins.

Suppose we use 49 \$2 coins. Since $\$2 \times 49 = \$98$, to reach a total of \$100, we would need two \$1 and no 25¢ coins, or one \$1 and four 25¢ coins, or no \$1 and eight 25¢ coins. Therefore, there are 3 different ways to make a total of \$100 if we use 49 \$2 coins.

Suppose we use 48 \$2 coins. Since $\$2 \times 48 = \$96$, to reach a total of \$100, we would need four \$1 and no 25¢ coins, or three \$1 and four 25¢ coins, or two \$1 and eight 25¢ coins, or one \$1 and twelve 25¢ coins, or no \$1 and sixteen 25¢ coins. Therefore, there are 5 different ways to make a total of \$100 if we use 48 \$2 coins.

We start to see a pattern. When we reduce the number of \$2 coins by one, the number of possible combinations using that many \$2 coins increases by 2. This is because there are 2 more options for the number of \$1 coins we can use. Thus, when we use 47 \$2 coins, there are 7 possible ways to make a total of \$100. When we use 46 \$2 coins, there are 9 possible ways to make a total of \$100, and so on. When we use 1 \$2 coin, there are 99 different ways to make the difference of \$98 (because you can use 0 to 98 \$1 coins). When we don't use any \$2 coins, there are 101 different ways to make a total of \$100 (because you can use 0 to 100 \$1 coins). Thus, the number of different combinations of coins that have a total value of \$100 is

$$1 + 3 + 5 + 7 + 9 + \cdots + 99 + 101$$

Adding and subtracting the even numbers from 2 to 100, we get

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \cdots + 98 + 99 + 100 + 101 - (2 + 4 + 6 + 8 + \cdots + 98 + 100)$$

Factoring out a factor of 2 from the subtracted even numbers, we get

$$(1 + 2 + 3 + \cdots + 100 + 101) - 2(1 + 2 + 3 + 4 + \cdots + 50)$$

We can then use the formula for the sum of the first $n$ positive integers to find that this expression is equal to

$$\frac{101(102)}{2} - 2\left(\frac{50(51)}{2}\right) = 101(51) - 50(51) = 2601$$

Therefore, there are 2601 different combinations of toonies, loonies, and/or quarters that have a total value of \$100.

**Extension:**

Let's look at the end of the previous computation another way.

$$\begin{aligned} 1+3+5+7+9+\cdots+99+101 &= \frac{101(102)}{2} - 2\left(\frac{50(51)}{2}\right) \\ &= 101(51) - 50(51) \\ &= 51(101-50) \\ &= 51(51) \\ &= 51^2 \end{aligned}$$

How many odd integers are in the list from 1 to 101? From 1 to 101, there are 101 integers. This list contains the even integers, from 2 to 100, which are 50 in total. Therefore, there are $101 - 50 = 51$ odd integers from 1 to 101.

Is it a coincidence that the sum of the first 51 odd positive integers is equal to $51^2$? Is the sum of the first 1000 odd positive integers equal to $1000^2$? Is the sum of the first $n$ odd positive integers equal to $n^2$?

We will develop a formula for the sum of the first $n$ odd positive integers.

We saw in the problem statement that the sum of the first $n$ positive integers is

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

Every odd positive integer can be written in the form $2n-1$, where $n$ is an integer $\geq 1$. When $n=1,\ 2n-1 = 2(1)-1 = 1$; when $n=2,\ 2n-1 = 2(2)-1 = 3$, and so on. So the $51^{\text{st}}$ odd positive integer is $2(51)-1 = 101$, as we determined above. The $n^{\text{th}}$ odd positive integer is $2n-1$. Let's consider the sum of the first $n$ odd positive integers. That is,

$$1+3+5+7+\cdots+(2n-3)+(2n-1)$$

Adding and subtracting the even numbers from 2 to $2n$, we get

$$\begin{aligned} &1+2+3+4+5+\cdots+(2n-3)+(2n-2)+(2n-1)+2n-(2+4+6+\cdots+(2n-2)+2n) \\ &=(1+2+3+4+\cdots+2n)-(2+4+6+8+\cdots+(2n-2)+2n) \end{aligned}$$

Factoring out a 2 from the subtracted even numbers, we get

$$(1+2+3+4+\cdots+2n)-2(1+2+3+\cdots+n)$$

We can then use the formula for the sum of the first $n$ positive integers to find that this expression is equal to

$$\begin{aligned} \frac{2n(2n+1)}{2} - 2\left(\frac{n(n+1)}{2}\right) &= n(2n+1) - n(n+1) \\ &= 2n^2 + n - n^2 - n \\ &= n^2 \end{aligned}$$

Therefore, the sum of the first $n$ odd positive integers is equal to $n^2$.

**For Further Thought:** Can you develop a formula for the sum of the first $n$ even positive integers?