Problem of the Week
Problem E and Solution
Coin Combinations

Problem
In Canada, a $2 coin is called a toonie, a $1 coin is called a loonie, and a 25¢ coin is called a quarter. Four quarters have a value of $1.

How many different combinations of toonies, loonies, and/or quarters have a total value of $100?

Solution
We will break the solution into cases based on the number of $2 coins used. For each case, we will count the number of possibilities for the number of $1 and 25¢ coins.

The maximum number of $2 coins we can use is 50, since $2 \times 50 = $100. If we use 50 $2 coins, then we do not need any $1 or 25¢ coins. Therefore, there is only one way to make a total of $100 if there are 50 $2 coins.

Suppose we use 49 $2 coins. Since $2 \times 49 = $98, to reach a total of $100, we would need two $1 and no 25¢ coins, or one $1 and four 25¢ coins, or no $1 and eight 25¢ coins. Therefore, there are 3 different ways to make a total of $100 if we use 49 $2 coins.

Suppose we use 48 $2 coins. Since $2 \times 48 = $96, to reach a total of $100, we would need four $1 and no 25¢ coins, or three $1 and four 25¢ coins, or two $1 and eight 25¢ coins, or one $1 and twelve 25¢ coins, or no $1 and sixteen 25¢ coins. Therefore, there are 5 different ways to make a total of $100 if we use 48 $2 coins.

We start to see a pattern. When we reduce the number of $2 coins by one, the number of possible combinations using that many $2 coins increases by 2. This is because there are 2 more options for the number of $1 coins we can use. Thus, when we use 47 $2 coins, there are 7 possible ways to make a total of $100. When we use 46 $2 coins, there are 9 possible ways to make a total of $100, and so on. When we use 1 $2 coin, there are 99 different ways to make the difference of $98 (because you can use 0 to 98 $1 coins). When we don’t use any $2 coins, there are 101 different ways to make a total of $100 (because you can use 0 to 100 $1 coins).

Thus, the number of different combinations of coins that have a total value of $100 is

\[1 + 3 + 5 + 7 + 9 + \cdots + 99 + 101\]

Adding and subtracting the even numbers from 2 to 100, we get

\[1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + \cdots + 98 + 99 + 100 + 101 = 2 + 4 + 6 + 8 + \cdots + 98 + 100\]

Factoring out a factor of 2 from the subtracted even numbers, we get

\[(1 + 2 + 3 + \cdots + 100 + 101) - 2(1 + 2 + 3 + 4 + \cdots + 50)\]

We can then use the formula for the sum of the first \(n\) positive integers to find that this expression is equal to

\[\frac{101(102)}{2} - 2 \left( \frac{50(51)}{2} \right) = 101(51) - 50(51) = 2601\]
Therefore, there are 2601 different combinations of toonies, loonies, and/or quarters that have a total value of $100.

**Extension:**

Let’s look at the end of the previous computation another way.

\[
1 + 3 + 5 + 7 + 9 + \cdots + 99 + 101 = \frac{101(102)}{2} - 2 \left( \frac{50(51)}{2} \right) \\
= 101(51) - 50(51) \\
= 51(101 - 50) \\
= 51(51) \\
= 51^2
\]

How many odd integers are in the list from 1 to 101? From 1 to 101, there are 101 integers. This list contains the even integers, from 2 to 100, which are 50 in total. Therefore, there are 101 − 50 = 51 odd integers from 1 to 101.

Is it a coincidence that the sum of the first 51 odd positive integers is equal to 51^2? Is the sum of the first 1000 odd positive integers equal to 1000^2? Is the sum of the first \( n \) odd positive integers equal to \( n^2 \)?

We will develop a formula for the sum of the first \( n \) odd positive integers.

We saw in the problem statement that the sum of the first \( n \) positive integers is

\[
1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}
\]

Every odd positive integer can be written in the form \( 2n - 1 \), where \( n \) is an integer \( \geq 1 \). When \( n = 1 \), \( 2n - 1 = 2(1) - 1 = 1 \); when \( n = 2 \), \( 2n - 1 = 2(2) - 1 = 3 \), and so on. So the 51st odd positive integer is \( 2(51) - 1 = 101 \), as we determined above. The \( n \)th odd positive integer is \( 2n - 1 \). Let’s consider the sum of the first \( n \) odd positive integers. That is,

\[
1 + 3 + 5 + 7 + \cdots + (2n - 3) + (2n - 1)
\]

Adding and subtracting the even numbers from 2 to \( 2n \), we get

\[
1 + 2 + 3 + 4 + 5 + \cdots + (2n - 3) + (2n - 2) + (2n - 1) + 2n - (2 + 4 + 6 + \cdots + (2n - 2) + 2n)
= (1 + 2 + 3 + 4 + \cdots + 2n) - (2 + 4 + 6 + 8 + \cdots + (2n - 2) + 2n)
\]

Factoring out a 2 from the subtracted even numbers, we get

\[
(1 + 2 + 3 + 4 + \cdots + 2n) - 2(1 + 2 + 3 + \cdots + n)
\]

We can then use the formula for the sum of the first \( n \) positive integers to find that this expression is equal to

\[
\frac{2n(2n + 1)}{2} - 2 \left( \frac{n(n + 1)}{2} \right) = n(2n + 1) - n(n + 1)
= 2n^2 + n - n^2 - n
= n^2
\]

Therefore, the sum of the first \( n \) odd positive integers is equal to \( n^2 \).

**For Further Thought:** Can you develop a formula for the sum of the first \( n \) even positive integers?