Problem of the Week
Problem E and Solution
Diagonal Distance

Problem
Square $ABCD$ has $K$ on $BC$, $L$ on $DC$, $M$ on $AD$, and $N$ on $AB$ such that $KLMN$ forms a rectangle, $\triangle AMN$ and $\triangle LKC$ are congruent isosceles triangles, and also $\triangle MDL$ and $\triangle BNK$ are congruent isosceles triangles. If the total area of the four triangles is 50 cm$^2$, what is the length of $MK$?

Solution
Let $x$ represent the lengths of the equal sides of $\triangle AMN$ and $\triangle LKC$, and let $y$ represent the lengths of the equal sides of $\triangle MDL$ and $\triangle BNK$.

Thus, area $\triangle AMN = \text{area } \triangle LKC = \frac{1}{2}x^2$, and area $\triangle MDL = \text{area } \triangle BNK = \frac{1}{2}y^2$.

Therefore, the total area of the four triangles is equal to $\frac{1}{2}x^2 + \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}y^2 = x^2 + y^2$.

Since we’re given that this area is 50 cm$^2$, we have $x^2 + y^2 = 50$.

Three different solutions to find the length of $MK$ are provided.

Solution 1
In $\triangle AMN$, $MN^2 = AM^2 + AN^2 = x^2 + x^2$, and in $\triangle BNK$, $NK^2 = BN^2 + BK^2 = y^2 + y^2$.

Since $MK$ is a diagonal of rectangle $KLMN$, then by the Pythagorean Theorem we have

$$MK^2 = MN^2 + NK^2 = x^2 + x^2 + y^2 + y^2 = x^2 + y^2 + x^2 + y^2 = 50 + 50 = 100$$

Since $MK > 0$, we have $MK = 10$ cm.
Solution 2

In $\triangle AMN$, $MN^2 = x^2 + x^2 = 2x^2$. Therefore, $MN = \sqrt{2}x$, since $x > 0$.
In $\triangle BNK$, $NK^2 = y^2 + y^2 = 2y^2$. Therefore, $NK = \sqrt{2}y$, since $y > 0$.

Since $MK$ is a diagonal of rectangle $KLMN$, then by the Pythagorean Theorem we have

$$MK^2 = MN^2 + NK^2$$
$$= (\sqrt{2}x)^2 + (\sqrt{2}y)^2$$
$$= 2x^2 + 2y^2$$
$$= 2(x^2 + y^2)$$
$$= 2(50)$$
$$= 100$$

Since $MK > 0$, we have $MK = 10$ cm.

Solution 3

We construct the line segment $KP$, where $P$ lies on $AD$ such that $KP$ is perpendicular to $AD$.

Then $APKB$ is a rectangle. Furthermore, $AP = BK = y$, $PK = AB = x + y$, and $PM = AM − AP = x − y$.

Since $\triangle PKM$ is a right-angled triangle, by the Pythagorean Theorem we have

$$MK^2 = PM^2 + PK^2$$
$$= (x − y)^2 + (x + y)^2$$
$$= x^2 − 2xy + y^2 + x^2 + 2xy + y^2$$
$$= 2x^2 + 2y^2$$
$$= 2(x^2 + y^2)$$
$$= 2(50)$$
$$= 100$$

Since $MK > 0$, we have $MK = 10$ cm.