Problem of the Week
Problem E and Solution
Fill ALL the Squares

Problem
Twelve squares are placed in a row forming the grid below. Each square is to be filled with an integer. After the third square, each integer in a square is the sum of the previous three integers. If we know the third integer is 6, the sixth integer is 11, and the eleventh integer is 14, determine all of the integers in the grid.

Solution
Let $a_1$ be the first integer in the grid, $a_2$ be the second, $a_3$ be the third, $a_4$ be the fourth, and so on, until $a_{12}$ which is the twelfth integer in the grid. We are given that $a_3 = 6$, $a_6 = 11$, and $a_{11} = 14$.

Each integer after the third integer is equal to the sum of the previous three integers. Therefore, $a_6 = a_3 + a_4 + a_5$. Thus, $11 = 6 + a_4 + a_5$ or $a_4 + a_5 = 5$.

We also have $a_7 = a_4 + a_5 + a_6 = a_4 + a_5 + 11 = 5 + 11 = 16$, since $a_4 + a_5 = 5$.

Similarly, $a_9 = a_6 + a_7 + a_8 = 11 + 16 + a_8 = 27 + a_8$, $a_{10} = a_7 + a_8 + a_9 = 16 + a_8 + a_9 = 16 + (a_8 + 27) = 2a_8 + 43$, and $a_{11} = a_8 + a_9 + a_{10} = (a_8) + (a_8 + 27) + (2a_8 + 43) = 4a_8 + 70$.

We are given that $a_{11} = 14$. Therefore, $4a_8 + 70 = 14$, or $4a_8 = -56$, or $a_8 = -14$.

Therefore, $a_9 = a_8 + 27 = -14 + 27 = 13$, and $a_{10} = 2a_8 + 43 = 2(-14) + 43 = 15$.

Also, $a_{12} = a_9 + a_{10} + a_{11} = 13 + 15 + 14 = 42$.

So far, we know that the integers in the grid, from left to right are $a_1, a_2, 6, a_4, a_5, 11, 16, -14, 13, 15, 14, 42$.

Working backwards, $-14 = a_5 + 11 + 16$, so $a_5 = -41$.

From earlier, $a_4 + a_5 = 5$. Since $a_5 = -41$, we know $a_4 = 46$.

Continuing working backwards, $a_5 = a_2 + 6 + a_4$, so $-41 = a_2 + 6 + 46$, or $a_2 = -93$.

Finally, $a_4 = a_1 + a_2 + 6$, so $46 = a_1 + (-93) + 6$, or $a_1 = 133$.

Therefore, the twelve integers in the grid, from left to right, are $133, -93, 6, 46, -41, 11, 16, -14, 13, 15, 14, 42$.

The filled grid is shown below. We can indeed check that each integer after the first three integers is equal to the sum of the previous three integers.

| 133 | -93 | 6 | 46 | -41 | 11 | 16 | -14 | 13 | 15 | 14 | 42 |