Problem of the Week
Problem E and Solution
Missing the Fives III

Problem

Bobbi lists the positive integers, in order, excluding all multiples of 5. Her resulting list is

$$1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, \dots$$

Determine the sum of the first 2023 integers in Bobbi's list.

Solution

Solution 1

We begin by determining which integers are in Bobbi's list. In each group of 5 consecutive integers beginning at 1, Bobbi lists 4 of the integers, since she leaves out each integer that is a multiple of 5. That is, in each of these groups of 5 integers, Bobbi's list contains $\frac{4}{5}$ of the integers.

Consider the positive integers from 1 to n, where n is a multiple of 5. Of these n integers, Bobbi's list contains $\frac{4}{5}n$ integers. Since Bobbi's list contains 2023 integers, which is not a multiple of 4, and 2024 is a multiple of 4, we will determine the sum of the first 2024 integers in Bobbi's list and then subtract the 2024th integer.

Now, $\frac{4}{5}n = 2024$ or $n = \frac{2024\times 5}{4} = 2530$. So we need to determine the sum of the first 2530 positive integers with the integers that are multiples of 5 removed. That is, we need to determine the sum

$$1 + 2 + 3 + 4 + 6 + \cdots + 2524 + 2526 + 2527 + 2528 + 2529$$

We will proceed to determine this sum by first calculating the sum of the integers from 1 to 2530. We will then subtract from that sum the sum of the integers in this list that are multiples of 5. We will also need to remove 2529, which is the 2024th number in the list.

The sum of the integers from 1 to n is given by $\frac{n(n+1)}{2}$, and so the sum of the integers from 1 to 2530 is equal to $\frac{(2530)(2531)}{2} = 3\,201\,715$.

The sum of the multiples of 5 in this list, $5 + 10 + 15 + \cdots + 2520 + 2525 + 2530$, can be written as $5(1 + 2 + 3 + \cdots + 504 + 505 + 506)$.
This is equal to $5 \times \frac{(506)(507)}{2} = 641\,355$.

Therefore, the sum of the first 2023 integers in Bobbi's list is
$3\,201\,715 - 641\,355 - 2529 = 2\,557\,831$.

Solution 2

In this solution, we will find the sum of the first 2024 integers in Bobbi's list by pairing up the integers, and then subtract the 2024th integer in her list. From Solution 1, we know that the 2024th number in Bobbi's list is 2529. Thus, the sum of the first 2024 integers in Bobbi's list is

$$1 + 2 + 3 + 4 + 6 + \cdots + 2524 + 2526 + 2527 + 2528 + 2529$$

The sum of the first and last integers in this list is $1 + 2529 = 2530$.
The sum of the second integer and the second last integer is $2 + 2528 = 2530$.
The sum of the third integer and the third last integer is $3 + 2527 = 2530$.
We continue in this way moving toward the middle of the list. That is, we move one number to the right of the previous first number, and one number to the left of the previous second number. Doing so, we notice that

- when the first number in the new pair is one more than the previous first number, then the number it is paired with is one less than the previous second number, and
- when the first number in the new pair is two more than the previous first number (as is the case when a multiple of 5 is omitted), then the number it is paired with is two less than the previous second number.

That is, as we continue moving toward the middle of Bobbi's list, each pair will continue to have a sum equal to 2530. Since there are 2024 numbers in Bobbi's list, there are 1012 such pairs, each having a sum of 2530. Thus, if Bobbi lists the positive integers, in order, leaving out the integers that are multiples of 5, the sum of the first 2024 integers in her list is $1012 \times 2530 = 2\,560\,360$. However, this includes the 2024th integer in her list. Therefore, the sum of the first 2023 integers in Bobbi's list is $2\,560\,360 - 2529 = 2\,557\,831$.