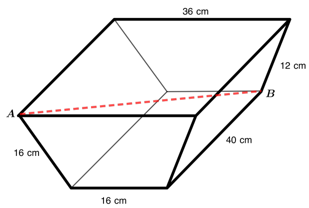


Problem of the Week
Problem E and Solution
Across the Prism

Problem

A trapezoidal prism is a prism in which opposite parallel ends are congruent trapezoids.

For the trapezoidal prism shown, the opposite parallel sides of each trapezoid are 36 cm and 16 cm in length. The non-parallel sides of each trapezoid are 16 cm and 12 cm in length. The prism is 40 cm long. The volume of the prism is 9984 cm^3.

A body diagonal of a prism is a line connecting two vertices that are not on the same face. Find the length of the body diagonal AB (indicated in the diagram), accurate to 1 decimal place.

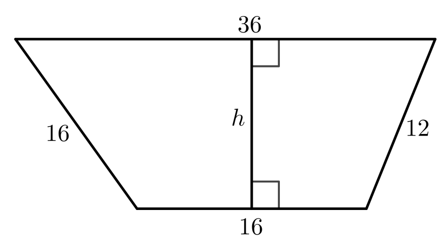
Solution

Let h represent the height of the trapezoid. We will determine the height using two different methods.

Method 1: Finding the Height Using the Given Volume

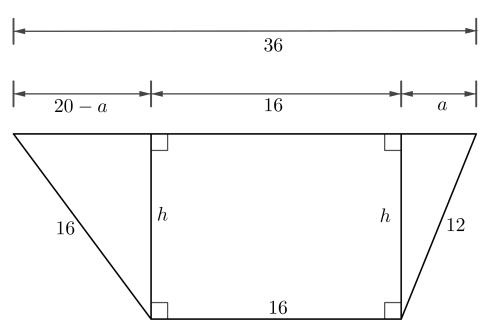
To find the volume, V, of a prism, we multiply the area of one of the congruent bases by the perpendicular distance, d, between the two bases. Since the bases are trapezoids, we can calculate the area of the base using the formula $A = \frac{h(a+b)}{2}$, where h is the perpendicular distance between the two parallel sides a and b. We know $V = 9984$ cm^3, $d = 40$ cm, $a = 16$ cm, and $b = 36$ cm. We can find h.

$$\begin{aligned} V &= \frac{h(a+b)}{2} \times d \\ 9984 &= \frac{h(16+36)}{2} \times 40 \\ 9984 &= 26h \times 40 \\ 9984 &= 1040h \\ h &= 9.6 \text{ cm} \end{aligned}$$



Method 2: Finding the Height Without Using the Given Volume

We start by breaking the trapezoids into two right-angled triangles and a rectangle. This can be done by drawing a line from each of the two vertices on the shorter parallel side to meet the longer parallel side at a right angle. The longer parallel side of the trapezoid breaks into pieces with lengths a cm, 16 cm, and $36 - 16 - a = 20 - a$ cm. This is shown in the diagram.



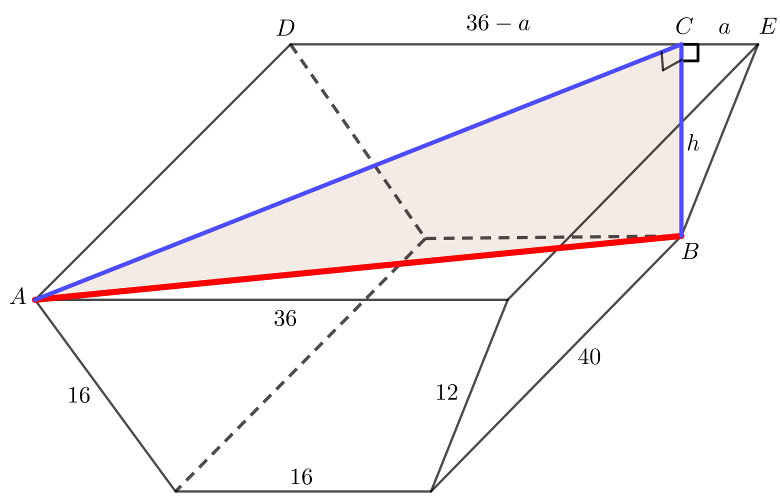
Using the Pythagorean Theorem, we can find two different expressions for h:
$h = \sqrt{16^2 - (20-a)^2}$ and $h = \sqrt{12^2 - a^2}$.

Since $h = h$, $\sqrt{16^2 - (20-a)^2} = \sqrt{12^2 - a^2}$.

Squaring both sides and expanding, we get $256 - 400 + 40a - a^2 = 144 - a^2$.
Simplifying further, we get $40a = 288$, and so $a = 7.2$ cm.

Substituting $a = 7.2$ into $h = \sqrt{144 - a^2}$, we find that $h = \sqrt{144 - 7.2^2} = 9.6$ cm.

With both methods, we find that $h = 9.6$ cm. We now want to find the length of AB. Let DE represent the length of the longer parallel side of the back trapezoid, with C on DE such that $DC \perp BC$. It follows that $BC = h = 9.6$ cm.



From Method 2, we know that $CE = a = 7.2$ cm. Then $DC = DE - CE = 36 - 7.2 = 28.8$ cm.

The sides of the prism are perpendicular to the ends, so $AD \perp DC$ and $\triangle CDA$ is a right-angled triangle. Using the Pythagorean Theorem in $\triangle CDA$,

$$\begin{aligned} AC^2 &= DC^2 + AD^2 \\ &= 28.8^2 + 40^2 \\ &= 2429.44 \end{aligned}$$

To find the required length, AB, we note that AC is a line segment drawn across the top of the prism. BC is a line segment perpendicular to the top and bottom of the prism. It follows that $\angle ACB = 90^\circ$ and $\triangle ACB$ is a right-angled triangle. We will use the Pythagorean Theorem in $\triangle ACB$ to find the length AB.

$$\begin{aligned} AB^2 &= AC^2 + BC^2 \\ &= 2429.44 + 9.6^2 \\ &= 2521.6 \end{aligned}$$

Since $AB > 0$, then we have $AB = \sqrt{2521.6} \approx 50.2$ cm.

Therefore, length of the body diagonal, AB, is approximately 50.2 cm.