Problem of the Week
Problem E and Solution
A Square in a Triangle

Problem
In \(\triangle ABC\), there is a right angle at \(B\) and the length of \(BC\) is twice the length of \(AB\). In other words, \(BC = 2AB\). Square \(DEFB\) is drawn inside \(\triangle ABC\) so that vertex \(D\) is somewhere on \(AB\) between \(A\) and \(B\), vertex \(E\) is somewhere on \(AC\) between \(A\) and \(C\), vertex \(F\) is somewhere on \(BC\) between \(B\) and \(C\), and the final vertex is at \(B\).

Square \(DEFB\) is called an inscribed square. Determine the ratio of the area of the inscribed square \(DEFB\) to the area of \(\triangle ABC\).

Solution
First we draw square \(DEFB\) according to the instructions in the problem. Let \(DB = BF = FE = ED = a\) and \(AD = b\). Since \(BC = 2AB\), it follows that \(BC = 2(AD + DB) = 2(a + b) = 2a + 2b\). Since \(BC = BF + FC\), it follows that \(2a + 2b = a + FC\), so \(FC = a + 2b\).

From here we present two solutions. In Solution 1, we solve the problem using similar triangles. In Solution 2, we place the diagram on the \(xy\)-plane and solve the problem using analytic geometry.

Solution 1
Consider \(\triangle ADE\) and \(\triangle ABC\). We will first show that \(\triangle ADE \sim \triangle ABC\).

Since \(DEFB\) is a square, then \(\angle EDB = 90^\circ\), and so \(\angle EDA = 180^\circ - \angle EDB = 180^\circ - 90^\circ = 90^\circ\). Therefore, \(\angle EDA = \angle ABC\). Also, \(\angle DAE = \angle BAC\) since they represent the same angle. Since the angles in a triangle add to \(180^\circ\), then we must also have \(\angle AED = \angle ACB\).

So \(\triangle ADE \sim \triangle ABC\), by Angle-Angle-Angle Triangle Similarity.

Since \(\triangle ADE \sim \triangle ABC\), then corresponding side lengths are in the same ratio. In particular,

\[
\frac{AD}{DE} = \frac{AB}{BC} \quad \frac{AD}{AB} = \frac{DE}{2AB} \quad \frac{b}{1} = \frac{1}{2} \quad a = 2b
\]

Since \(BC = 2a + 2b\) and \(a = 2b\), then \(BC = 2(2b) + 2b = 6b\). Since \(AB = a + b\) and \(a = 2b\), then \(AB = 2b + b = 3b\). The area of \(\triangle ABC\) is \(\frac{1}{2}(BC \times AB) = \frac{1}{2}(6b \times 3b) = 9b^2\).

The area of square \(DEFB\) is \(a \times a = a^2 = (2b)^2 = 4b^2\).

The ratio of the area of inscribed square \(DEFB\) to the area of \(\triangle ABC\) is \(4b^2 : 9b^2 = 4 : 9\), since \(b > 0\).
Solution 2

First we place the triangle on the $xy$-plane with $B$ at $(0, 0)$ and $BC$ along the positive $x$-axis. The coordinates of $D$ are $(0, a)$, the coordinates of $A$ are $(0, a + b)$, the coordinates of $F$ are $(a, 0)$, the coordinates of $E$ are $(a, a)$, and the coordinates of $C$ are $(2a + 2b, 0)$.

Let’s determine the equation of the line through $A$, $E$, and $C$.

Since this line passes through $(0, a + b)$, then we know it has $y$-intercept $a + b$.

Since it passes through $(0, a + b)$ and $(a, a)$, then the line has slope $\frac{a-(a+b)}{a-0} = -\frac{b}{a}$.

Therefore, the equation of the line through $A$, $E$, and $C$ is $y = \left(-\frac{b}{a}\right)x + a + b$.

Since $C(2a + 2b, 0)$ lies on this line, then substituting $x = 2a + 2b$ and $y = 0$ into $y = \left(-\frac{b}{a}\right)x + a + b$ gives

\[
0 = \left(-\frac{b}{a}\right)(2a + 2b) + a + b \\
0 = (-b)(2a + 2b) + (a)(a + b) \\
0 = -2ab - 2b^2 + a^2 + ab \\
0 = a^2 - ab - 2b^2 \\
0 = (a + b)(a - 2b)
\]

Thus, $a = -b$ or $a = 2b$. But since $a, b > 0$, then $a = -b$ is inadmissible and we must have $a = 2b$.

Since $BC = 2a + 2b$ and $a = 2b$, then $BC = 2(2b) + 2b = 6b$. Since $AB = a + b$ and $a = 2b$, then $AB = 2b + b = 3b$. The area of $\triangle ABC$ is $\frac{1}{2}(BC \times AB) = \frac{1}{2}(6b \times 3b) = 9b^2$.

The area of square $DEFB$ is $a \times a = a^2 = (2b)^2 = 4b^2$.

The ratio of the area of inscribed square $DEFB$ to the area of $\triangle ABC$ is $4b^2 : 9b^2 = 4 : 9$, since $b > 0$.

Note:

From the equation $0 = (-b)(2a + 2b) + (a)(a + b)$, we could have instead factored $(2a + 2b)$ to obtain $0 = (-2b)(a + b) + a(a + b)$. Since $a, b > 0$, $a + b > 0$, so we could have divided out the common factor of $(a + b)$ leaving $0 = -2b + a$ which simplifies to $a = 2b$. Thus, the factoring of $a^2 - ab - 2b^2$ to determine $a = 2b$ would not have been necessary.

Extension:

If, in the original problem, $BC = kAB$, where $k > 0$, and the square was inscribed as given, what would be the ratio of the area of square $DEFB$ to the area of $\triangle ABC$?