Problem of the Week
Problem E and Solution
The Angle Between

Problem
Rectangle $PQRS$ has $PQ = 3$ and $QR = 4$. Points $T$ and $U$ are on side $PS$ such that $PT = US = 1$. Determine the measure of $\angle TQU$, in degrees and rounded to 1 decimal place.

Solution
Let $X$ be the point on $QR$ such that $UX$ is parallel to $SR$. Then $\angle UXQ = 90^\circ$. Also, $UX = SR = 3$, $XR = US = 1$, and therefore $QX = 3$. It follows that $\triangle UXQ$ is an isosceles right-angled triangle, and so $\angle UQX = \angle QUX = 45^\circ$. From here we present three different solutions.

Solution 1
Since $\triangle TPQ$ is a right-angled triangle, $\tan(\angle TQP) = \frac{1}{3}$, and so $\angle TQP \approx 18.4^\circ$. Since $\angle PQX = 90^\circ$, we can calculate the value of $\angle TQU$ as follows.

$$
\angle PQX = \angle PQT + \angle TQU + \angle UQX
$$

$$
\angle TQU = \angle PQX - \angle PQT - \angle UQX
$$

$$
\angle TQU \approx 90^\circ - 18.4^\circ - 45^\circ
$$

$$
\angle TQU \approx 26.6^\circ
$$

Therefore, $\angle TQU \approx 26.6^\circ$.

Solution 2
Since $\triangle TPQ$ is a right-angled triangle, by the Pythagorean Theorem, $TQ^2 = PT^2 + PQ^2 = 1^2 + 3^2 = 10$. Therefore $TQ = \sqrt{10}$, since $TQ > 0$.

Since $PQRS$ is a rectangle, $PS = QR = 4$, and $PT = US = 1$, it follows that $TU = 2$. Since $\triangle UXQ$ is a right-angled triangle, by the Pythagorean Theorem, $QU^2 = UX^2 + QX^2 = 3^2 + 3^2 = 18$. Therefore $QU = \sqrt{18}$, since $QU > 0$. Now we will use the cosine law in $\triangle TQU$. 

Above we present three different solutions.
\[
TU^2 = TQ^2 + QU^2 - 2(TQ)(QU)\cos(\angle TQU)
\]
\[
2^2 = 10 + 18 - 2(\sqrt{10})(\sqrt{18})\cos(\angle TQU)
\]
\[
4 - 10 - 18 = -2\sqrt{10}\sqrt{18}\cos(\angle TQU)
\]
\[
-24 = -2\sqrt{10}\sqrt{18}\cos(\angle TQU)
\]
\[
\frac{12}{\sqrt{10}\sqrt{18}} = \cos(\angle TQU)
\]
\[
\angle TQU \approx 26.6^\circ
\]

Therefore, \(\angle TQU \approx 26.6^\circ\).

**Solution 3**

Since \(\triangle TPQ\) is a right-angled triangle, by the Pythagorean Theorem,
\[
TQ^2 = PT^2 + PQ^2 = 1^2 + 3^2 = 10.
\]
Therefore \(TQ = \sqrt{10}\), since \(TQ > 0\).

Since \(PQRS\) is a rectangle, \(PS = QR = 4\), and \(PT = US = 1\), it follows that \(TU = 2\).

Since \(UX\) is parallel to \(SR\), then \(\angle PUX = 90^\circ\). Since \(\angle QUX = 45^\circ\), it follows that \(\angle TUQ = 45^\circ\).

Now we will use the sine law in \(\triangle TQU\).
\[
\frac{\sin(\angle TQU)}{TU} = \frac{\sin(\angle TUQ)}{TQ}
\]
\[
\sin(\angle TQU) = \frac{\sin 45^\circ}{\sqrt{10}}
\]
\[
\angle TQU \approx 26.6^\circ
\]

Therefore, \(\angle TQU \approx 26.6^\circ\).