Problem of the Week

Problem E and Solution

Sum Product Function

Problem
A function, \( g \), has \( g(2) = 5 \) and \( g(3) = 7 \). In addition, \( g \) has the property that

\[ g(a) + g(b) = g(ab) \]

for all positive integers \( a \) and \( b \).

For example, \( g(6) = g(2) + g(3) = 12 \).

What is the value of \( g(648) \)?

Solution
We can rewrite \( g(648) \) as:

\[
g(648) = g(2 \cdot 324) \\
= g(2) + g(324) \\
= g(2) + g(2 \cdot 162) \\
= g(2) + g(2) + g(162) \\
= g(2) + g(2) + g(2 \cdot 81) \\
= g(2) + g(2) + g(2) + g(81) \\
= g(2) + g(2) + g(2) + g(3 \cdot 27) \\
= g(2) + g(2) + g(2) + g(3) + g(27) \\
= g(2) + g(2) + g(2) + g(3) + g(3 \cdot 9) \\
= g(2) + g(2) + g(2) + g(3) + g(3) + g(9) \\
= g(2) + g(2) + g(2) + g(3) + g(3) + g(3 \cdot 3) \\
= g(2) + g(2) + g(2) + g(3) + g(3) + g(3) + g(3)
\]

Therefore, \( g(648) = 3g(2) + 4g(3) = 3(5) + 4(7) = 43 \).

Note:
While this answers the question, is there actually a function that satisfies the requirements? The answer is yes.

One function that satisfies the requirements of the problem is the function \( g \) defined by

\[ g(1) = 0 \text{ and } g(2^p3^q r) = 5p + 7q \]

for all non-negative integers \( p \) and \( q \) and all positive integers \( r \) that are not divisible by 2 or by 3. Can you see why this function satisfies the requirements?