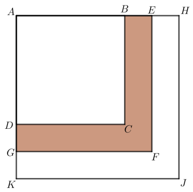


Problem of the Week

Problem E and Solution

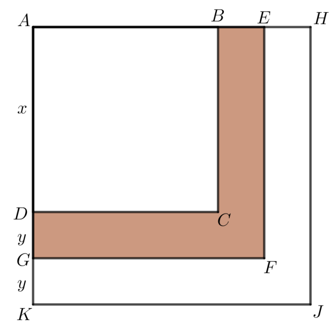
Three Squares

Problem

The three squares $ABCD$, $AEFG$, and $AHJK$ overlap as shown in the diagram.

The side length of each square, in centimetres, is a positive integer. The area of square $AEFG$ that is not covered by square $ABCD$ is 33 cm^2. That is, the area of the shaded region $BEFGDC$ is 33 cm^2. If $DG = GK$, determine all possible side lengths of each square.

Solution



Let $AD = x$ cm and $DG = y$ cm. Therefore $GK = DG = y$ cm. Also, since the side length of each square is an integer, x and y are integers.
The shaded region has area 33 cm^2. The shaded region is equal to the area of the square with side length AG minus the area of the square with side length AD.

Since $AD = x$ and $AG = AD + DG = x + y$, we have

$$\begin{aligned} 33 &= (\text{area of square with side length } AG) - (\text{area of square with side length } AD) \\ &= (x+y)^2 - x^2 \\ &= x^2 + 2xy + y^2 - x^2 \\ &= 2xy + y^2 \\ &= y(2x+y) \end{aligned}$$

Since x and y are integers, so is $2x + y$. Therefore, $2x + y$ and y are two positive integers that multiply to give 33. Therefore, we must have $y = 1$ and $2x + y = 33$, or $y = 3$ and $2x + y = 11$, or $y = 11$ and $2x + y = 3$, or $y = 33$ and $2x + y = 1$. The last two would imply that $x < 0$, which is not possible. Therefore, $y = 1$ and $2x + y = 33$, or $y = 3$ and $2x + y = 11$.

When $y = 1$ and $2x + y = 33$, it follows that $x = 16$. Then square $ABCD$ has side length $x = 16$ cm, square $AEFG$ has side length $x + y = 17$ cm, and square $AHJK$ has side length $x + 2y = 18$ cm.

When $y = 3$ and $2x + y = 11$, it follows that $x = 4$. Then square $ABCD$ has side length $x = 4$ cm, square $AEFG$ has side length $x + y = 7$ cm, and square $AHJK$ has side length $x + 2y = 10$ cm.

Therefore, there are two possible sets of squares. The squares are either 16 cm $\times$ 16 cm and 17 cm $\times$ 17 cm and 18 cm $\times$ 18 cm, or 4 cm $\times$ 4 cm and 7 cm $\times$ 7 cm and 10 cm $\times$ 10 cm. Each of these sets of squares satisfies the conditions of the problem.