Problem of the Week
Problem D and Solution
Find the Largest Area

Problem
Rectangle $ACEG$ has $B$ on $AC$ and $F$ on $EG$ such that $BF$ is parallel to $CE$. Also, $D$ is on $CE$ and $H$ is on $AG$ such that $HD$ is parallel to $AC$, and $BF$ intersects $HD$ at $J$. The area of rectangle $ABJH$ is $6 \text{ cm}^2$ and the area of rectangle $JDEF$ is $15 \text{ cm}^2$.

If the dimensions of rectangles $ABJH$ and $JDEF$, in centimetres, are integers, then determine the largest possible area of rectangle $ACEG$.

Solution
Let $AB = x$, $AH = y$, $JD = a$ and $JF = b$.

Then,

- $HJ = GF = AB = x$
- $BJ = CD = AH = y$
- $BC = FE = JD = a$
- $HG = DE = JF = b$

Thus, we have

\[
\text{area}(ACEG) = \text{area}(ABJH) + \text{area}(BCDJ) + \text{area}(JDEF) + \text{area}(HJFG)
\]

\[
= 6 + ya + 15 + xb
\]

\[
= 21 + ya + xb
\]

Since the area of rectangle $ABJH$ is $6 \text{ cm}^2$ and the side lengths of $ABJH$ are integers, then the side lengths must be 1 and 6 or 2 and 3. That is, $x = 1 \text{ cm}$ and $y = 6 \text{ cm}$, $x = 6 \text{ cm}$ and $y = 1 \text{ cm}$, $x = 2 \text{ cm}$ and $y = 3 \text{ cm}$, or $x = 3 \text{ cm}$ and $y = 2 \text{ cm}$.

Since the area of rectangle $JDEF$ is $15 \text{ cm}^2$ and the side lengths of $JDEF$ are integers, then the side lengths must be 1 and 15 or 3 and 5. That is, $a = 1 \text{ cm}$ and $b = 15 \text{ cm}$, $a = 15 \text{ cm}$ and $b = 1 \text{ cm}$, $a = 3 \text{ cm}$ and $b = 5 \text{ cm}$, or $a = 5 \text{ cm}$ and $b = 3 \text{ cm}$.

To maximize the area, we need to pick the values of $x$, $y$, $a$, and $b$ which make $ya + xb$ as large as possible. We will now break into cases based on the possible side lengths of $ABJH$ and $JDEF$ and calculate the area of $ACEG$ in each case. We do not need to try all 16 possible pairings, because trying $x = 1 \text{ cm}$ and $y = 6 \text{ cm}$ with the four possibilities of $a$ and $b$ will give the same 4 areas, in some order, as trying $x = 6 \text{ cm}$ and $y = 1 \text{ cm}$ with the four possibilities of $a$ and $b$. Similarly, trying $x = 2 \text{ cm}$ and $y = 3 \text{ cm}$ with the four possibilities of $a$ and $b$ will give the same 4 areas, in some order, as trying $x = 3 \text{ cm}$ and $y = 2 \text{ cm}$ with the four possibilities of $a$ and $b$. (As an extension, we will leave it to you to think about why this is the case.)
• **Case 1:** $x = 1$ cm, $y = 6$ cm, $a = 1$ cm, $b = 15$ cm
  
  Then $\text{area}(ACEG) = 21 + ya + xb = 21 + 6(1) + 1(15) = 42$ cm$^2$.

• **Case 2:** $x = 1$ cm, $y = 6$ cm, $a = 15$ cm, $b = 1$ cm
  
  Then $\text{area}(ACEG) = 21 + ya + xb = 21 + 6(15) + 1(1) = 112$ cm$^2$.

• **Case 3:** $x = 1$ cm, $y = 6$ cm, $a = 3$ cm, $b = 5$ cm
  
  Then $\text{area}(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44$ cm$^2$.

• **Case 4:** $x = 1$ cm, $y = 6$ cm, $a = 5$ cm, $b = 3$ cm
  
  Then $\text{area}(ACEG) = 21 + ya + xb = 21 + 6(3) + 1(5) = 44$ cm$^2$.

• **Case 5:** $x = 2$ cm, $y = 3$ cm, $a = 1$, $b = 15$ cm
  
  Then $\text{area}(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54$ cm$^2$.

• **Case 6:** $x = 2$ cm, $y = 3$ cm, $a = 15$, $b = 1$ cm
  
  Then $\text{area}(ACEG) = 21 + ya + xb = 21 + 3(1) + 2(15) = 54$ cm$^2$.

• **Case 7:** $x = 2$ cm, $y = 3$ cm, $a = 3$, $b = 5$ cm
  
  Then $\text{area}(ACEG) = 21 + ya + xb = 21 + 3(3) + 2(5) = 40$ cm$^2$.

• **Case 8:** $x = 2$ cm, $y = 3$ cm, $a = 5$, $b = 3$ cm
  
  Then $\text{area}(ACEG) = 21 + ya + xb = 21 + 3(5) + 2(3) = 42$ cm$^2$.

We see that the maximum area is $112$ cm$^2$, and occurs when $x = 1$ cm, $y = 6$ cm and $a = 15$ cm, $b = 1$ cm. It will also occur when $x = 6$ cm, $y = 1$ cm and $a = 1$ cm, $b = 15$ cm.

The following diagrams show the calculated values placed on the original diagram. The diagram given in the problem was definitely not drawn to scale! Both solutions produce rectangles with dimensions 7 cm by 16 cm, and area 112 cm$^2$. 