Problem of the Week
Problem C and Solution
All Equal

Problem
Using two cuts, we want to divide the 6 m by 6 m grid shown into three regions of equal area.

One way to do so is by making a horizontal cut through $H$ and a second horizontal cut through $K$. This method of cutting the grid works, but is not very creative.

To make things a little more interesting, we must still make two straight cuts, but each cut must start at point $P$. Each of these two cuts will pass through a point on the outer perimeter of the grid.

Find the length of each cut. Round your answer to one decimal.

Solution
The area of the entire 6 m by 6 m square grid is $6 \times 6 = 36 \text{ m}^2$. Since the square is divided into three regions of equal area, the area of each region must be $\frac{36}{3} = 12 \text{ m}^2$.

Consider the line through $P$ that passes through some point on side $QM$. Let $A$ be the point where this line intersects $QM$.

Since $\angle PMQ = 90^\circ$, $\triangle PMA$ is a right-angled triangle with base $PM = 6 \text{ m}$ and height $MA$.

Using the formula $\text{area} = \frac{\text{base} \times \text{height}}{2}$, we have area of $\triangle PMA = \frac{6 \times MA}{2} = 3 \times MA$.

We need the area of $\triangle PMA$ to be $12 \text{ m}^2$. Therefore, $3 \times MA = 12$, and so $MA = 4 \text{ m}$. Since $H$ is the point on $QM$ with $MH = 4 \text{ m}$, we must have $A = H$. Therefore, one line passes through the point $H$.

Since $\triangle PMA$ is a right-angled triangle, using the Pythagorean Theorem we have

$$PA^2 = PM^2 + MA^2$$
$$= 6^2 + 4^2$$
$$= 36 + 16$$
$$= 52$$

Therefore, $PA = \sqrt{52} \approx 7.2$, since $PA > 0$. 
Consider the line through $P$ that passes through some point on side $RQ$. Let $B$ be the point where this line intersects $RQ$.

Since $\angle PRQ = 90^\circ$, $\triangle PRB$ is a right-angled triangle with height $PR = 6$ m and base $RB$.

Using the formula $\text{area} = \frac{\text{base} \times \text{height}}{2}$, we have $\text{area of } \triangle PRB = \frac{RB \times 6}{2} = 3 \times RB$.

We need the area of $\triangle PRB$ to be $12$ m$^2$. Therefore, $3 \times RB = 12$, and so $RB = 4$ m. Since $V$ is the point on $RQ$ with $RV = 4$ m, we must have $B = V$. Therefore, the other line passes through the point $V$.

Therefore, one line passes through point $H$ and the other passes through point $V$.

Since $\triangle PRB$ is a right-angled triangle, using the Pythagorean Theorem we have

$$PB^2 = PR^2 + RB^2$$
$$= 6^2 + 4^2$$
$$= 36 + 16$$
$$= 52$$

Therefore, $PB = \sqrt{52} \approx 7.2$, since $PB > 0$.

Therefore, the length of each cut is approximately 7.2 m.

**Extension:**

Try dividing the grid into three regions of equal area using three cuts. (Each cut does not necessarily need to be to the outer perimeter of the grid.)