Problem of the Week
Problem C and Solution
Three Perimeters

Problem
A median is a line segment drawn from the vertex of a triangle to the midpoint of the opposite side.

In \( \triangle DEF \), a median is drawn from vertex \( D \), meeting side \( EF \) at point \( M \). The perimeter of \( \triangle DEF \) is 24. The perimeter of \( \triangle DEM \) is 18. The perimeter \( \triangle DFM \) is 16. Determine the length of the median \( DM \).

Solution
Solution 1
The perimeter of a triangle is equal to the sum of its three side lengths. Notice that the length of side \( EF \) is equal to the sum of the lengths of sides \( EM \) and \( MF \). It follows that when we combine the perimeters of \( \triangle DEM \) and \( \triangle DFM \), we obtain the perimeter of \( \triangle DEF \) plus two lengths of the median \( DM \).

\[
t + p + m = 18
\]

In other words, since the perimeter of \( \triangle DEM \) is 18, the perimeter of \( \triangle DFM \) is 16, and the perimeter of \( \triangle DEF \) is 24, it follows that \( 18 + 16 = 24 + 2 \times DM \). Then \( 34 = 24 + 2 \times DM \), and so \( 2 \times DM = 10 \). Therefore, the length of the median \( DM \) is 5.

Solution 2
In this solution, we take a more algebraic approach to solving the problem, using more formal equation solving.

Let \( DE = t \), \( EM = p \), \( MF = q \), \( DF = r \), and \( DM = m \).

Since the perimeter of \( \triangle DEM \) is 18, we can write the following equation.

\[
t + p + m = 18
\]
\[
t + p = 18 - m
\] (1)
Since the perimeter of $\triangle DFM$ is 16, we can write the following equation.

\[ q + r + m = 16 \]
\[ q + r = 16 - m \]  \hspace{1cm} (2)

Since the perimeter of $\triangle DEF$ is 24, we can write the following equation.

\[ t + p + q + r = 24 \]  \hspace{1cm} (3)

Adding equations (1) and (2) gives the following.

\[ t + p = 18 - m \]  \hspace{1cm} (1)
\[ q + r = 16 - m \]  \hspace{1cm} (2)
\[ t + p + q + r = 18 - m + 16 - m \]

However from equation (3), we know that $t + p + q + r = 24$. So we can write and solve the following equation.

\[ 18 - m + 16 - m = 24 \]
\[ 34 - 2m = 24 \]
\[ -2m = 24 - 34 \]
\[ -2m = -10 \]
\[ \frac{-2m}{-2} = \frac{-10}{-2} \]
\[ m = 5 \]

Therefore, the length of the median $DM$ is 5.

**Extension:**

In the solution we never used the fact that $DM$ is a median and that $EM = MF$. This means that there could be other triangles that satisfy the conditions of the problem without $DM$ being the median. Indeed there are! Try creating a few different triangles with $DM = 5$ that satisfy all the conditions of the problem except the condition that $DM$ is a median. You can do this using manipulatives, geometry software, or by hand. However, you may need some high school mathematics to calculate the precise dimensions.

It turns out that there is only one triangle that satisfies all the conditions of the problem including the fact that $DM$ is a median.