Problem of the Week
Problem E and Solution
Odd Sum

Problem

A sequence consists of 2022 terms. Each term after the first term is 1 greater than the previous term. The sum of the 2022 terms is 31 341.

Determine the sum of the terms in the odd-numbered positions. That is, determine the sum of every second term starting with the first term and ending with the second last term.

NOTE:
In solving the above problem, it may be helpful to use the fact that the sum of the first n positive integers is equal to $\frac{n(n+1)}{2}$. That is,

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

Solution

Solution 1

Let t_n denote the n^{th} term in the sequence.

Let S_O represent the sum of the terms in the odd-numbered positions. That is,

$$S_O = t_1 + t_3 + t_5 + \cdots + t_{2021}$$

Let S_E represent the sum of the terms in the even-numbered positions. That is,

$$S_E = t_2 + t_4 + t_6 + \cdots + t_{2022}$$

Since there are 2022 terms, and half of the terms of the sequence are in even-numbered positions and half are in odd-numbered positions, there are 1011 terms in S_O and 1011 terms in S_E.

Let S represent the sum of the 2022 terms. That is,

$$S = S_O + S_E = 31\,341 \qquad (1)$$

Since each term after the first term is 1 greater than the term before,

$$t_2 = t_1 + 1$$
$$t_4 = t_3 + 1$$
$$t_6 = t_5 + 1$$

and so on, until

$$t_{2022} = t_{2021} + 1$$

Now,

$$\begin{aligned} S_E &= t_2 + t_4 + t_6 + \cdots + t_{2016} + t_{2022} \\ &= (t_1 + 1) + (t_3 + 1) + (t_5 + 1) + \cdots + (t_{2019} + 1) + (t_{2021} + 1) \\ &= (t_1 + t_3 + t_5 + \cdots + t_{2019} + t_{2021}) + 1011 \\ &= S_O + 1011 \end{aligned}$$

Substituting $S_E = S_O + 1011$ into equation (1),

$$\begin{aligned} S_O + S_E &= 31\,341 \\ S_O + S_O + 1011 &= 31\,341 \\ 2S_O &= 31\,341 - 1011 \\ 2S_O &= 30\,330 \\ S_O &= 15\,165 \end{aligned}$$

Therefore, the sum of the terms in the odd-numbered positions is $15\,165$.

Notice that this solution did not need the formula given in the note after the problem.

Solution 2

Let t_1 represent the first term in the sequence. Every term in the sequence can be written in terms of t_1. The second term is 1 more than the first term, the third term is 2 more than the first term, the fourth term is 3 more than the first term, and so on. Thus,

$$\begin{aligned} t_1 + t_2 + t_3 + t_4 + \cdots + t_{2021} + t_{2022} &= 31\,341 \\ t_1 + (t_1 + 1) + (t_1 + 2) + (t_1 + 3) + \cdots + (t_1 + 2020) + (t_1 + 2021) &= 31\,341 \\ 2022t_1 + (1 + 2 + 3 + \cdots + 2020 + 2021) &= 31\,341 \end{aligned}$$

Using the formula for the sum of the first n positive integers with $n = 2021$,

$$2022t_1 + \frac{2021(2022)}{2} = 34\,341$$

Dividing by 2022,

$$\begin{aligned} t_1 + \frac{2021}{2} &= \frac{31\,341}{2022} \\ t_1 + 1010.5 &= 15.5 \\ t_1 &= -995 \end{aligned}$$

Since the first term in the sequence is -995, we know that the original sum is

$$\begin{aligned} &t_1 + t_2 + t_3 + t_4 + \cdots + t_{2020} + t_{2021} + t_{2022} \\ &\qquad = t_1 + (t_1 + 1) + (t_1 + 2) + (t_1 + 3) + \cdots + (t_1 + 2020) + (t_1 + 2021) \\ &\qquad = -995 - 994 - 993 - 992 - \cdots + 1025 + 1026 \end{aligned}$$

We are interested in the sum of the terms in the odd-numbered positions. That is, we're interested in the sum

$$-995 - 993 - 991 - \cdots + 991 + 993 + 995 + 997 + 999 + \cdots + 1023 + 1025$$

From this point, we will present two different methods for determining this sum.

- *Method 1:*

  Notice that this sum includes all of the odd integers from -995 to 995, inclusive. This sum is 0. Thus,

$$\begin{aligned} -995 - 993 - 991 - \cdots + 991 + 993 + 995 + 997 + 999 + \cdots + 1023 + 1025 \\ &= 0 + 997 + 999 + \cdots + 1023 + 1025 \\ &= 997(15) + 2 + 4 + \cdots + 28 \\ &= 14\,955 + 2(1 + 2 + \cdots + 14) \\ &= 14\,955 + 2\left[\frac{14(15)}{2}\right] \\ &= 14\,955 + 210 \\ &= 15\,165 \end{aligned}$$

- *Method 2:*

  This is an arithmetic series with $n = 1011$ terms, first term $t_1 = -995$, and last term $t_n = 1025$.

  Then, using the formula $S_n = n\left[\frac{t_1 + t_n}{2}\right]$ for the sum of a series,

$$\begin{aligned} -995 - 993 - 991 - 989 - \cdots + 1023 + 1025 &= 1011\left[\frac{-995 + 1025}{2}\right] \\ &= 1011(15) \\ &= 15\,165 \end{aligned}$$

Therefore, the sum of the terms in the odd-numbered positions is $15\,165$.