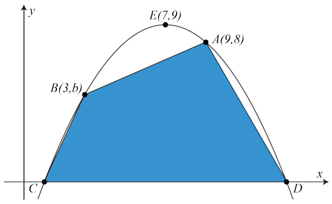


Problem of the Week
Problem E and Solution
Parabolic Art

Problem

Kenna likes making artistic creations using parabolas, to put on the walls of her math classroom. She drew a parabola with vertex $E(7,9)$ and plotted points $A(9,8)$ and $B(3,b)$ on the parabola as well as points C and D where the parabola intersects the x-axis, with C to the left of D. Then she connected points A, B, C, and D to form quadrilateral $ABCD$, and painted it blue. What is the area of quadrilateral $ABCD$?

Solution

First we need to find the equation of the parabola. Then, we can find the x-intercepts of the parabola and the y-coordinate of point B on the parabola.

We are given the vertex of the parabola, $E(7,9)$. Using the vertex form of the equation of a parabola, $y = a(x-h)^2 + k$, with vertex $(h,k) = (7,9)$, the equation of the parabola is $y = a(x-7)^2 + 9$.

Since the point $A(9,8)$ is on the parabola, we can substitute $(x,y) = (9,8)$ into the equation $y = a(x-7)^2 + 9$ to find the value of a.

$$\begin{aligned} 8 &= a(9-7)^2 + 9 \\ 8 &= a(4) + 9 \\ -1 &= 4a \\ -\frac{1}{4} &= a \end{aligned}$$

The equation of the parabola is therefore $y = -\frac{1}{4}(x-7)^2 + 9$.

To find the y-coordinate of $B(3,b)$, we substitute $(x,y) = (3,b)$ into the equation of the parabola.

$$\begin{aligned} b &= -\frac{1}{4}(3-7)^2 + 9 \\ &= -\frac{1}{4}(16) + 9 \\ &= -4 + 9 \\ &= 5 \end{aligned}$$

Therefore, the coordinates of B are $(3,5)$.

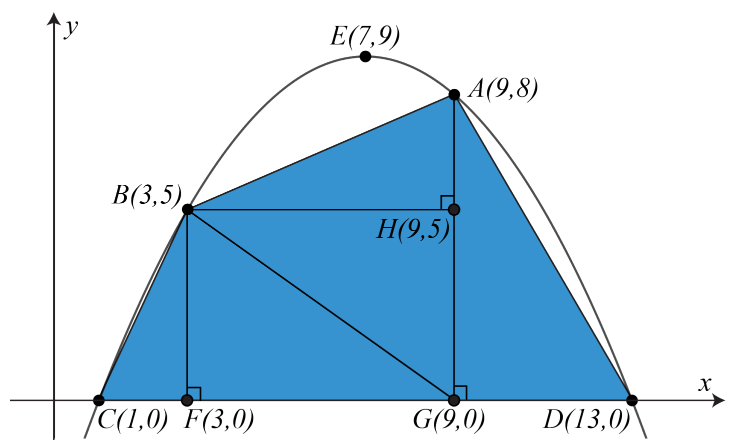
To find the x-intercepts of the parabola, we substitute $y = 0$ into the equation of the parabola.

$$\begin{aligned} 0 &= -\frac{1}{4}(x-7)^2 + 9 \\ -9 &= -\frac{1}{4}(x-7)^2 \\ 36 &= (x-7)^2 \\ \pm 6 &= x - 7 \end{aligned}$$

It follows that $x - 7 = -6$ or $x - 7 = 6$. Then the x-intercepts of the parabola are 1 and 13. Therefore, the coordinates of C and D are $C(1, 0)$ and $D(13, 0)$.

Now that we know the coordinates of A, B, C, and D, we can calculate the area of quadrilateral $ABCD$. There are many ways to do this. We will proceed as follows.

From $B(3, 5)$ and $A(9, 8)$, drop perpendiculars, intersecting the x-axis at $F(3, 0)$ and $G(9, 0)$, respectively. From $B(3, 5)$ draw a line perpendicular to AG, intersecting AG at $H(9, 5)$. Draw line segment BG.



Note that line segments BG and AG divide the quadrilateral into three regions: $\triangle CGB$, $\triangle AGD$, and $\triangle AGB$.

We will use the coordinates of the points to find the lengths of several horizontal and vertical line segments that will be required for the area calculation.

$BH = 9 - 3 = 6,\ CG = 9 - 1 = 8,\ GD = 13 - 9 = 4,\ BF = 5 - 0 = 5,$ and $AG = 8 - 0 = 8.$

To determine the area of $ABCD$, we will find the sum of the areas of $\triangle CGB$, $\triangle AGD$ and $\triangle AGB$.

$$\begin{aligned} \text{Area } ABCD &= \text{Area } \triangle CGB + \text{Area } \triangle AGD + \text{Area } \triangle AGB \\ &= \frac{CG \times BF}{2} + \frac{AG \times GD}{2} + \frac{AG \times BH}{2} \\ &= \frac{8 \times 5}{2} + \frac{8 \times 4}{2} + \frac{8 \times 6}{2} \\ &= 20 + 16 + 24 \\ &= 60 \text{ units}^2 \end{aligned}$$

Therefore, the area of quadrilateral $ABCD$ is 60 units^2.