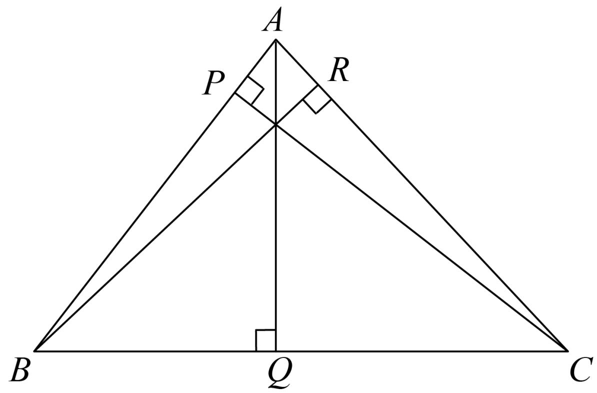
Problem of the Week

Problem E and Solution

Now I Know My ABCs

Problem

In triangle ABC, point P lies on AB, point Q lies on BC, and point R lies on AC such that AQ, BR, and CP are altitudes with lengths 21 cm, 24 cm, and 56 cm, respectively.



Determine the measure, in degrees, of $\angle ABC$, and the lengths, in centimetres, of AB, BC, and CA.

Note the diagram is not drawn to scale.

Solution

Let $BC = a$, $AC = b$, and $AB = c$. We will present two methods for determining that $21a = 24b = 56c$, and then continue on with the rest of the solution.

- *Method 1: Use Areas*

  We can find the area of $\triangle ABC$ by multiplying the length of the altitude (the height) by the corresponding base and dividing by 2. Therefore,

  $$\frac{AQ \times BC}{2} = \frac{BR \times AC}{2} = \frac{CP \times AB}{2}$$

  Substituting $AQ = 21$, $BR = 24$, and $CP = 56$, and multiplying through by 2 gives us $21a = 24b = 56c$.

- *Method 2: Use Trigonometry*

  In right-angled $\triangle ARB$, $\sin A = \frac{BR}{AB} = \frac{24}{c}$. In $\triangle APC$, $\sin A = \frac{CP}{AC} = \frac{56}{b}$. Putting these together gives $\frac{24}{c} = \frac{56}{b}$, or $24b = 56c$.

  In right-angled $\triangle BAQ$, $\sin B = \frac{AQ}{AB} = \frac{21}{c}$. In $\triangle BPC$, $\sin B = \frac{CP}{BC} = \frac{56}{a}$. Putting these together gives $\frac{21}{c} = \frac{56}{a}$, or $21a = 56c$.

  Combining these gives $21a = 24b = 56c$.

We now will continue on with the rest of the solution.

From $21a = 24b$ we obtain $b = \frac{21}{24}a = \frac{7}{8}a$, and from $21a = 56c$ we obtain $c = \frac{21}{56}a = \frac{3}{8}a$. The ratio of the sides in $\triangle ABC$ is therefore $a : b : c = a : \frac{7}{8}a : \frac{3}{8}a = 8 : 7 : 3$. Let $BC = 8x$, $AC = 7x$, and $AB = 3x$, where $x > 0$.

Using the cosine law,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 - 2(AB)(BC)\cos(\angle ABC) \\ (7x)^2 &= (3x)^2 + (8x)^2 - 2(3x)(8x)\cos(\angle ABC) \\ 49x^2 &= 9x^2 + 64x^2 - 48x^2\cos(\angle ABC) \end{aligned}$$

Since $x > 0$, we know $x^2 \neq 0$. So dividing by x^2,

$$49 = 73 - 48\cos(\angle ABC)$$

Rearranging,

$$\begin{aligned} 48\cos(\angle ABC) &= 24 \\ \cos(\angle ABC) &= \frac{1}{2} \end{aligned}$$

Therefore, $\angle ABC = 60^\circ$.

In right $\triangle BPC$,

$$\begin{aligned} \frac{CP}{BC} &= \sin 60^\circ \\ BC &= \frac{CP}{\sin 60^\circ} \\ BC &= \frac{56}{\frac{\sqrt{3}}{2}} \\ BC &= \frac{112}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} \\ BC &= \frac{112\sqrt{3}}{3} \end{aligned}$$

However, $BC = 8x$. Therefore,

$$\begin{aligned} 8x &= \frac{112\sqrt{3}}{3} \\ x &= \frac{14\sqrt{3}}{3} \\ 3x &= 14\sqrt{3} \\ 7x &= \frac{98\sqrt{3}}{3} \end{aligned}$$

Therefore, $\angle ABC = 60^\circ$, and the side lengths of $\triangle ABC$ are $AB = 3x = 14\sqrt{3}$ cm, $AC = 7x = \dfrac{98\sqrt{3}}{3}$ cm, and $BC = \dfrac{112\sqrt{3}}{3}$ cm.