



## Problem of the Week

### Problem E and Solution

#### Let's All Ride a Bike

#### Problem

Grayson's Groupcycles rents bikes with multiple seats for large groups of people. They rent 7-seater, 13-seater, and 25-seater bikes. A group of 14 people could fit on two 7-seater bikes, however a group of 15 people could not fit exactly on any of the bikes since no combination of bikes have exactly 15 seats.

What is the largest group size that *cannot* fit exactly on any combination of bikes from Grayson's Groupcycles?

#### Solution

Any group size that is a multiple of 7 can fit on multiple 7-seater bikes. Similarly, any group size that is a multiple of 13 can fit on multiple 13-seater bikes, and any group size that is a multiple of 25 can fit on multiple 25-seater bikes. These multiples are listed below.

multiples of 7 : 7, 14, 21, 28, 35, 42, 49, ...

multiples of 13 : 13, 26, 39, 52, ...

multiples of 25 : 25, 50, 75, ...

Putting these numbers together, along with sums of the different multiples, gives us the following list of group sizes that can fit exactly on some combination of bikes from Grayson's Groupcycles.

7, 13, 14, 20 ( $= 7 + 13$ ), 21, 25, 26, 27 ( $= 14 + 13$ ), 28, 32 ( $= 7 + 25$ ), 33 ( $= 7 + 26$ ),  
34 ( $= 21 + 13$ ), 35, 38 ( $= 13 + 25$ ), 39, 40 ( $= 14 + 26$ ), 41 ( $= 28 + 13$ ), 42,  
45 ( $= 7 + 13 + 25$ ), 46 ( $= 21 + 25$ ), 47 ( $= 21 + 26$ ), 48 ( $= 35 + 13$ ), 49, 50, 51 ( $= 26 + 25$ )

The missing numbers from the above list correspond to the group sizes that cannot fit exactly on any combination of bikes. The largest of these group sizes appears to be 44, however we must justify that this is the maximum group size that cannot fit exactly on any combination of bikes. To do this, we note that group sizes of 45, 46, 47, 48, 49, 50, and 51 can all fit exactly on some combination of bikes. This corresponds to 7 consecutive group sizes. If we add 7 to each of these group sizes, the additional 7 people could fit on a 7-seater bike. It follows that group sizes of 52, 53, 54, 55, 56, 57, and 58 can also fit exactly on some combination of bikes. In this way, we can continue to add 7 to each of these group sizes to obtain the next set of 7 consecutive group sizes and determine that they can also fit exactly on some combination of bikes. It follows that every group size of 45 or more can fit exactly on some combination of bikes.

Thus, the largest group size that cannot fit exactly on any combination of bikes from Grayson's Groupcycles is 44.

**Extension:** It turns out there are only 26 group sizes that cannot fit exactly on any combination of bikes. If Grayson's Groupcycles added a 3-seater bike, how many group sizes wouldn't be able to fit exactly on any combination of bikes?