Problem of the Week
Problem E and Solution
Medians

Problem
In $\triangle ABC$, $\angle ABC = 90^\circ$. A median is drawn from $A$ to side $BC$, meeting $BC$ at $M$ such that $AM = 5$. A second median is drawn from $C$ to side $AB$, meeting $AB$ at $N$ such that $CN = 2\sqrt{10}$.

Determine the length of the longest side of $\triangle ABC$.

Note: In a triangle, a median is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.

Solution
Since $AM$ is a median, $M$ is the midpoint of $BC$. Then $BM = MC$. Let $BM = MC = y$.
Since $CN$ is a median, $N$ is the midpoint of $AB$. Then $AN = NB$. Let $AN = NB = x$.

Since $\angle B = 90^\circ$, $\triangle NBC$ is a right-angled triangle. Using the Pythagorean Theorem,

$\begin{align*}
NB^2 + BC^2 &= CN^2 \\
x^2 + (2y)^2 &= (2\sqrt{10})^2 \\
x^2 + 4y^2 &= 40 \tag{1}
\end{align*}$

Since $\angle B = 90^\circ$, $\triangle ABM$ is a right-angled triangle. Using the Pythagorean Theorem,

$\begin{align*}
AB^2 + BM^2 &= AM^2 \\
(2x)^2 + y^2 &= 5^2 \\
4x^2 + y^2 &= 25 \tag{2}
\end{align*}$

Adding equations (1) and (2), we get $5x^2 + 5y^2 = 65$ or $x^2 + y^2 = 13$.

The longest side of $\triangle ABC$ is the hypotenuse $AC$. Using the Pythagorean Theorem,

$\begin{align*}
AC^2 &= AB^2 + BC^2 \\
&= (2x)^2 + (2y)^2 \\
&= 4x^2 + 4y^2 \\
&= 4(x^2 + y^2)
\end{align*}$

Since $x^2 + y^2 = 13$, we have $AC^2 = 4(13)$. And since $AC > 0$, $AC = 2\sqrt{13}$ follows.
Therefore, the length of the longest side of $\triangle ABC$ is $2\sqrt{13}$.

Note: The solver could have instead solved a system of equations to find $x = 2$ and $y = 3$, and then proceed to solve for the longest side. The above approach was provided to expose the solver to alternate way to think about the solution of this problem.