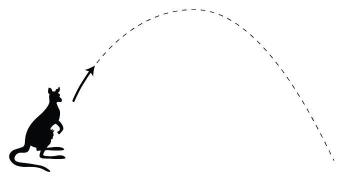

Problem of the Week

Problem E and Solution

Reach for the Sky

Problem

The equation $y = -5x^2 + ax + b$, where a and b are real numbers and $a \neq b$, represents a parabola. If this parabola passes through the points with coordinates (a, b) and (b, a), determine the maximum value of the parabola.

Solution

Since (a, b) lies on the parabola, it satisfies the equation of the parabola. We can substitute $x = a$ and $y = b$ into the equation $y = -5x^2 + ax + b$.

$$\begin{aligned} b &= -5a^2 + a^2 + b \\ b &= -4a^2 + b \\ 0 &= -4a^2 \\ 0 &= a^2 \\ 0 &= a \end{aligned}$$

The equation becomes $y = -5x^2 + 0x + b$, or simply $y = -5x^2 + b$.

Since (b, a) lies on the parabola, it satisfies the equation of the parabola. We can substitute $x = b$ and $y = a = 0$ into the equation $y = -5x^2 + b$.

$$\begin{aligned} 0 &= -5b^2 + b \\ 0 &= b(-5b + 1) \end{aligned}$$

This means that $b = 0$ or $-5b + 1 = 0$. Therefore, $b = 0$ or $b = \frac{1}{5}$.
Since $a \neq b$ and $a = 0$, then $b = 0$ is inadmissible.

Therefore, $b = \frac{1}{5}$ and the equation representing the parabola is $y = -5x^2 + \frac{1}{5}$. The parabola opens down and the vertex of the parabola is $\left(0, \frac{1}{5}\right)$, and so the maximum value of the parabola is $\frac{1}{5}$.