Problem of the Week
Problem E and Solution
Fine Line

Problem

Suppose that $f(x) = ax + b$ and $g(x) = f^{-1}(x)$ for all values of x. That is, g is the inverse of the function f.

If $f(x) - g(x) = 2022$ for all values of x, determine all possible values for a and b.

Solution

Since $f(x) = ax + b$, we can determine an expression for $g(x) = f^{-1}(x)$ by letting $y = f(x)$ to obtain $y = ax + b$. We then interchange x and y to obtain $x = ay + b$, which we solve for y to obtain $ay = x - b$ or $y = \dfrac{x}{a} - \dfrac{b}{a}$.

Therefore, $f^{-1}(x) = \dfrac{x}{a} - \dfrac{b}{a}$. Note that $a \neq 0$. (This makes sense since the function $f(x) = b$ has a graph which is a horizontal line, and so cannot be invertible.)

Therefore, the equation $f(x) - g(x) = 2022$ becomes

$$\begin{aligned}(ax + b) - \left(\frac{x}{a} - \frac{b}{a}\right) &= 2022 \\ \left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) &= 2022\end{aligned}$$

This is true for all x.

From here, we will present two approaches for determining the possible values for a and b.

- **Approach 1:** Comparing coefficients

  Since the equation
  $$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 2022 = 0x + 2022$$
  is true for all x, then the coefficients of the linear expression on the left side must match the coefficients of the linear expression on the right side.

  Therefore, $a - \dfrac{1}{a} = 0$ and $b + \dfrac{b}{a} = 2022$.

  From the first of these equations, we obtain $a = \dfrac{1}{a}$ or $a^2 = 1$, which gives $a = 1$ or $a = -1$.

  If $a = 1$, the equation $b + \dfrac{b}{a} = 2022$ becomes $b + b = 2022$, which gives $b = 1011$.

  If $a = -1$ the equation $b + \dfrac{b}{a} = 2022$ becomes $b - b = 2022$ which is not possible.

  Therefore, we must have $a = 1$ and $b = 1011$, and so $f(x) = x + 1011$.

- **Approach 2:** Trying specific values for x

  Since the equation
  $$\left(a - \frac{1}{a}\right)x + \left(b + \frac{b}{a}\right) = 2022$$
  is true for all values of x, then it must be true for any specific values of x that we choose.

  Choosing $x = b$, we obtain
  $$\begin{aligned}\left(a - \frac{1}{a}\right)b + \left(b + \frac{b}{a}\right) &= 2022 \\ ab + b &= 2022 \qquad (1)\end{aligned}$$

  Choosing $x = 0$, we obtain
  $$\begin{aligned}0 + b + \frac{b}{a} &= 2022 \\ b + \frac{b}{a} &= 2022 \\ \frac{ab+b}{a} &= 2022\end{aligned}$$

  Then, substituting $ab + b = 2022$ from equation (1), we obtain
  $$\begin{aligned}\frac{ab+b}{a} &= 2022 \\ \frac{2022}{a} &= 2022 \\ a &= 1\end{aligned}$$

  Since $a = 1$, then $ab + b = 2022$ gives $2b = 2022$ or $b = 1011$.

  Thus, $f(x) = x + 1011$.

In summary, the only possible values for a and b for which the given equation is true for all x are $a = 1$ and $b = 1011$.