$t_1 + t_2 + t_3 + t_4 + t_5 + \cdots$

Problem of the Week
Problem E and Solution
Just Sum Numbers

Problem

Kaori writes a sequence with the property that after the first two terms in the sequence, each term is equal to one more than the term before it, minus the term before that. In other words, $t_n = 1 + t_{n-1} - t_{n-2}$, for $n \geq 3$, where t_n denotes the n^{th} term in the sequence.

The first term in Kaori's sequence is x and the second term is y, where x and y are real numbers. That is, $t_1 = x$ and $t_2 = y$. Determine the sum of the first 2021 terms in her sequence, as an expression in terms of x and y.

Solution

We are given that $t_1 = x$, $t_2 = y$, and $t_n = 1 + t_{n-1} - t_{n-2}$, for $n \geq 3$.
Let's use the equation $t_n = 1 + t_{n-1} - t_{n-2}$ for $n \geq 3$.

$$\begin{aligned}
t_3 &= 1 + t_2 - t_1 = 1 + y - x \\
t_4 &= 1 + t_3 - t_2 = 1 + (1 + y - x) - y = 2 - x \\
t_5 &= 1 + t_4 - t_3 = 1 + (2 - x) - (1 + y - x) = 2 - y \\
t_6 &= 1 + t_5 - t_4 = 1 + (2 - y) - (2 - x) = 1 - y + x \\
t_7 &= 1 + t_6 - t_5 = 1 + (1 - y + x) - (2 - y) = x \\
t_8 &= 1 + t_7 - t_6 = 1 + x - (1 - y + x) = y
\end{aligned}$$

Since $t_7 = t_1$ and $t_8 = t_2$, and each term in the sequence depends only on the previous two terms, it follows that the sequence repeats every six terms.

The sum of the first six terms in the sequence is equal to
$x + y + (1 + y - x) + (2 - x) + (2 - y) + (1 - y + x) = 6$.
It follows that the sum of each successive group of six terms is also equal to 6.

We note that $2022 = 6 \times 337$, so the 2022^{nd} term of the sequence is the end of a group of six terms. Thus, the sum of the first 2022 terms in the sequence is equal to $6 \times 337 = 2022$. It also follows that $t_{2022} = t_6 = 1 - y + x$.

Since the sum of the first 2021 terms is equal to the sum of the first 2022 terms minus the 2022^{nd} term, we know that the sum of the first 2021 terms of the sequence is equal to $2022 - (1 - y + x) = 2021 + y - x$.