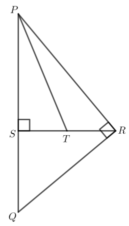


Problem of the Week
Problem D and Solution
Which Term is Which?

Problem

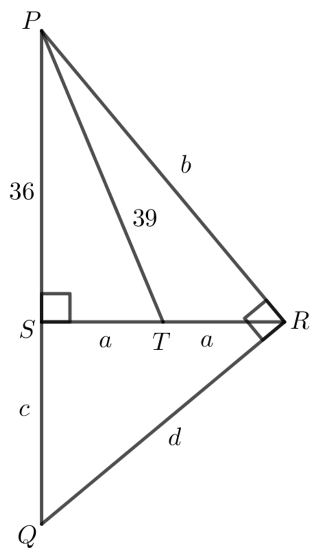
In $\triangle PQR$, $\angle PRQ = 90°$. An altitude is drawn in $\triangle PQR$ from R to PQ, intersecting PQ at S. A median is drawn in $\triangle PSR$ from P to SR, intersecting SR at T.

If the length of the median PT is 39 and the length of PS is 36, determine the length of QS.

NOTE: An *altitude* of a triangle is a line segment drawn from a vertex of the triangle perpendicular to the opposite side. A *median* is a line segment drawn from a vertex of the triangle to the midpoint of the opposite side.

Solution

Since T is a median in $\triangle PSR$, $ST = TR$. Let $ST = TR = a$. Let $PR = b$, $QS = c$, and $QR = d$. The variables and the given information, $PS = 36$ and $PT = 39$, are shown in the diagram.



Since $\triangle PST$ contains a right angle at S,

$$\begin{aligned} ST^2 &= PT^2 - PS^2 \\ a^2 &= 39^2 - 36^2 \\ &= 225 \end{aligned}$$

Then, since $a > 0$, $a = 15$ follows. Thus, $SR = 2a = 30$.

Since $\triangle PSR$ contains a right angle at S,

$$\begin{aligned} PR^2 &= PS^2 + SR^2 \\ b^2 &= 36^2 + 30^2 \\ &= 2196 \end{aligned}$$

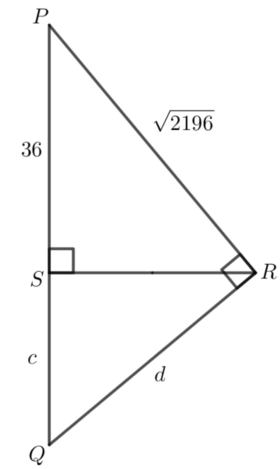
Then, since $b > 0$, $b = \sqrt{2196}$ follows.

We will now use $a = 15$ and $b = \sqrt{2196}$ in the three solutions that follow.

Solution 1

In $\triangle PSR$ and $\triangle PRQ$, $\angle PSR = \angle PRQ = 90^\circ$ and $\angle SPR = \angle QPR$, a common angle. So $\triangle PSR$ is similar to $\triangle PRQ$. It follows that

$$\begin{aligned} \frac{PS}{PR} &= \frac{PR}{PQ} \\ \frac{36}{\sqrt{2196}} &= \frac{\sqrt{2196}}{36+c} \\ 1296 + 36c &= 2196 \\ 36c &= 900 \\ c &= 25 \end{aligned}$$



Thus, the length of QS is 25.

Solution 2

Since $\triangle RSQ$ contains a right angle at S, $QR^2 = QS^2 + SR^2 = c^2 + 30^2 = c^2 + 900$. Therefore, $d^2 = c^2 + 900$.

Since $\triangle PQR$ contains a right angle at R, $PQ^2 = PR^2 + QR^2$. Therefore, $(36+c)^2 = (\sqrt{2196})^2 + d^2$, which simplifies to $1296 + 72c + c^2 = 2196 + d^2$. This further simplifies to $c^2 + 72c = 900 + d^2$.

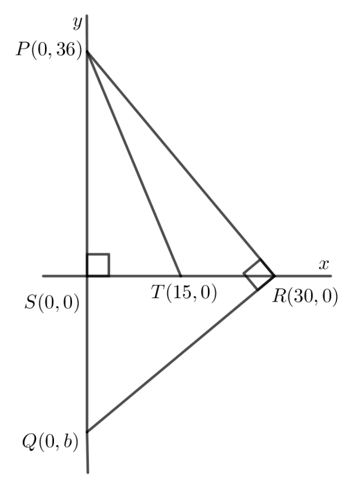
Substituting $d^2 = c^2 + 900$, we obtain $c^2 + 72c = 900 + c^2 + 900$. Simplifying, we get $72c = 1800$ and $c = 25$ follows.

Thus, the length of QS is 25.

Solution 3

Position $\triangle PQR$ on the xy-plane so that PQ lies along the y-axis, and altitude SR lies along the positive x-axis with S at the origin. Then P has coordinates $(0, 36)$, T has coordinates $(15, 0)$, and R has coordinates $(30, 0)$.
Since Q is on the y-axis, let Q have coordinates $(0, b)$ with $b < 0$.
Notice that

$$\text{slope } PR = \frac{36-0}{0-30} = \frac{-6}{5} \text{ and slope } QR = \frac{b-0}{0-30} = \frac{b}{-30}$$



Since $\angle PRQ = 90^\circ$, $PR \perp QR$, and so their slopes are negative reciprocals of each other. That is, $\frac{b}{-30} = \frac{5}{6}$, and so $b = -25$.

It then follows that the coordinates of Q are $(0, -25)$. Thus, the length of QS is 25.