$10^{2021} - 2021$

Problem of the Week
Problem D and Solution
An Exponential Year

Problem

Determine the sum of the digits in the difference when $10^{2021} - 2021$ is evaluated.

Solution

Solution 1

When the number 10^{2021} is written out, there is a one followed by 2021 zeroes, for a total of 2022 digits. Let's look at what happens in our effort to subtract.

$$\begin{array}{cccccccccccc}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & - & 2 & 0 & 2 & 1 \\
\hline
\end{array}$$

Using the standard subtraction algorithm, we start with the rightmost digits. In this case we need to borrow. But the borrowing creates a chain reaction. The result after the borrowing is complete is shown below.

$$\begin{array}{cccccccccccc}
\cancel{1} & \cancel{0}^{9} & \cancel{0}^{9} & \cancel{0}^{9} & \cancel{0}^{9} & \cdots & \cancel{0}^{9} & \cancel{0}^{9} & \cancel{0}^{9} & \cancel{0}^{9} & \cancel{0}^{9} & {}^{1}0 \\
 & & & & & & & - & 2 & 0 & 2 & 1 \\
\hline
 & 9 & 9 & 9 & 9 & \cdots & 9 & 9 & 7 & 9 & 7 & 9
\end{array}$$

The four rightmost digits in the difference are 7, 9, 7, and 9. To the left of these digits every digit is a 9. But how many nines are there? The difference has one less digit than 10^{2021}, so has 2021 digits. We have accounted for the four rightmost digits. So to the left of 7979 there are $2021 - 4 = 2017$ nines.

Therefore, the digit sum is

$$2017 \times 9 + (7 + 9 + 7 + 9) = 18\,153 + 32 = 18\,185.$$

Solution 2

The expression $10^{2021} - 2021$ has the same value as $(10^{2021} - 1) - (2021 - 1)$.

As mentioned in Solution 1, when 10^{2021} is written out, there is a one followed by 2021 zeroes, for a total of 2022 digits. The number $(10^{2021} - 1)$ is one less than 10^{2021} and therefore is the positive whole number made up of exactly 2021 nines. When 1 is subtracted from 2021, the difference is 2020. The following is the equivalent subtraction question:

$$\begin{array}{cccccccccccc}
 & 9 & 9 & 9 & 9 & \cdots & 9 & 9 & 9 & 9 & 9 & 9 \\
 & & & & & & & - & 2 & 0 & 2 & 0 \\
\hline
 & 9 & 9 & 9 & 9 & \cdots & 9 & 9 & 7 & 9 & 7 & 9
\end{array}$$

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