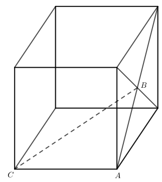


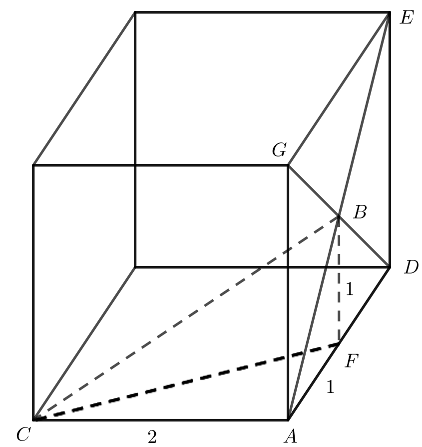
Problem of the Week
Problem D and Solution
To the Other Side

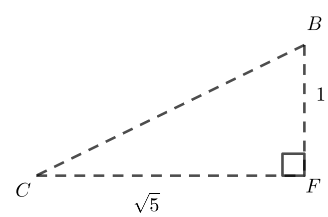
Problem
Points A and C are vertices of a cube with side length 2 cm, and B is the point of intersection of the diagonals of one face of the cube, as shown below. Determine the length of CB.

Solution

Solution 1

Label vertices D, E and G, as shown.
Drop a perpendicular from B to AD. Let F be the point where the perpendicular meets AD. Join B to F and C to F.
The faces of a cube are squares. The diagonals of a square meet at the centre of the square. Therefore, $BF = 1$ and $AF = 1$.
Now, $\triangle CAF$ is right-angled.
Using the Pythagorean Theorem in $\triangle CAF$,
$CF^2 = CA^2 + AF^2 = 2^2 + 1^2 = 5$.
Therefore, $CF = \sqrt{5}$, since $CF > 0$.
Looking at $\triangle CFB$, we know from above that $CF = \sqrt{5}$ and $BF = 1$. We also know that $\angle CFB = 90°$.





Because of the three-dimensional nature of the problem, it may not be obvious to all that $\angle CFB = 90°$. To help visualize this, notice that CF and BF lie along faces of the cube that meet at 90°.

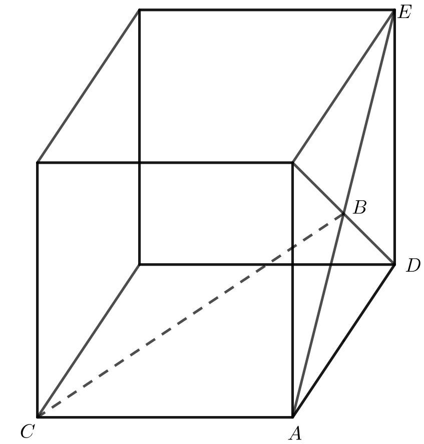
Using the Pythagorean Theorem in $\triangle CFB$,
$CB^2 = CF^2 + BF^2 = \sqrt{5}^2 + 1^2 = 5 + 1 = 6$. Since $CB > 0$, we have $CB = \sqrt{6}$.

Therefore, the length of CB is $\sqrt{6}$ cm.

There is a second solution on the next page.

Solution 2

Label vertices D and E, as shown.



The faces of a cube are squares. The diagonals of a square right bisect each other. It follows that $AB = BE = \frac{1}{2}AE$. Since the face is a square, $\angle ADE = 90^\circ$ and $\triangle ADE$ is right-angled. Using the Pythagorean Theorem in $\triangle ADE$, $AE^2 = AD^2 + DE^2 = 2^2 + 2^2 = 8$. Since $AE > 0$, we have $AE = \sqrt{8}$. Then $AB = \frac{1}{2}AE = \frac{\sqrt{8}}{2}$.

Because of the three-dimensional nature of the problem, it may not be obvious to all that $\angle CAB = 90^\circ$. To help visualize this, note that $\angle CAD = 90^\circ$ because the face of the cube is a square. Rotate AD counterclockwise about point A on the side face of the cube so that the image of AD lies along AB. The corner angle will not change as a result of the rotation, so $\angle CAD = \angle CAB = 90^\circ$.

We can now use the Pythagorean Theorem in $\triangle CAB$ to find the length CB.

$$CB^2 = CA^2 + AB^2 = 2^2 + \left(\frac{\sqrt{8}}{2}\right)^2 = 4 + \frac{8}{4} = 4 + 2 = 6$$

Since $CB > 0$, we have $CB = \sqrt{6}$ cm.

Therefore, the length of CB is $\sqrt{6}$ cm.

Note, we could have simplified $AB = \frac{1}{2}AE = \frac{\sqrt{8}}{2}$ to $\sqrt{2}$ as follows:

$$\frac{\sqrt{8}}{2} = \frac{\sqrt{4 \times 2}}{2} = \frac{\sqrt{4} \times \sqrt{2}}{2} = \frac{2\sqrt{2}}{2} = \sqrt{2}.$$

The calculation of CB would have been simpler using $AB = \sqrt{2}$. Often simplifying radicals is not a part of the curriculum at the grade 9 or 10 level.