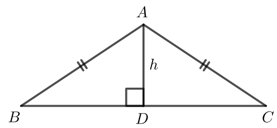


Problem of the Week

Problem D and Solution

Different Lengths

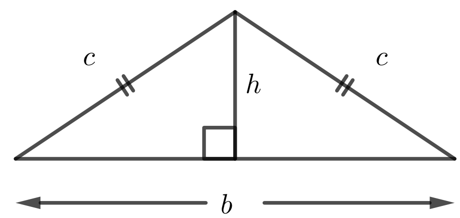
Problem

$\triangle ABC$ is isosceles with $AB = AC$. All three side lengths of $\triangle ABC$ and also altitude AD are positive integers.

If the area of $\triangle ABC$ is 60 cm^2, determine all possible perimeters of $\triangle ABC$.

Solution

Let the base of $\triangle ABC$ have length b and the equal sides have length c, as shown in the diagram to the right.



The area of $\triangle ABC$ is $\dfrac{\text{base} \times \text{height}}{2} = \dfrac{bh}{2}$.

Since this area is given to be 60 cm^2, we have $\dfrac{bh}{2} = 60$ or $bh = 120$.

We are given that b and h are positive integers. We will consider the positive factors of 120 to generate all possibilities for b and h. Since the altitude AD bisects BC, $\triangle ABC$ is composed of two congruent right-angled triangles, each with side lengths c, h, and $\frac{b}{2}$. We will use the Pythagorean Theorem in one of these right-angled triangles to generate a value of c for each possibility.

h	b	$\frac{b}{2}$	$c^2 = h^2 + (\frac{b}{2})^2$	Valid?
1	120	60	3601	No, c is not an integer
2	60	30	904	No, c is not an integer
3	40	20	409	No, c is not an integer
4	30	15	241	No, c is not an integer
5	24	12	169	Yes, $c = 13$
6	20	10	136	No, c is not an integer
8	15	7.5	120.25	No, c is not an integer
10	12	6	136	No, c is not an integer
12	10	5	169	Yes, $c = 13$
15	8	4	241	No, c is not an integer
20	6	3	409	No, c is not an integer
24	5	2.5	582.25	No, c is not an integer
30	4	2	904	No, c is not an integer
40	3	1.5	1602.25	No, c is not an integer
60	2	1	3601	No, c is not an integer
120	1	0.5	14 400.25	No, c is not an integer

We see that there are two solutions for (h, b, c). They are $(5, 24, 13)$ and $(12, 10, 13)$.

The side lengths of the corresponding triangles are 24, 13, and 13 and 10, 13, and 13.

Therefore, the perimeter of $\triangle ABC$ is either 50 cm or 36 cm.