



## Problem of the Week

### Problem D and Solution

#### Working on Camera

#### Problem

Today we are going to be working on camera. To be more precise, we are going to count certain arrangements of the letters in the word *CAMERA*.

The six letters, *C*, *A*, *M*, *E*, *R*, and *A* are arranged to form six letter “words”. When examining the “words”, how many of them have the vowels *A*, *A*, and *E* appearing in alphabetical order and the consonants *C*, *M*, and *R* not appearing in alphabetical order? The vowels may or may not be adjacent to each other and the consonants may or may not be adjacent to each other.

For example, each of *MAAERC* and *ARAEMC* are valid arrangements, but *ACAMER*, *MEAARC*, and *AEACMR* are invalid arrangements.

#### Solution

##### Solution 1

In this solution we will count all of the valid six letter arrangements of *C*, *A*, *M*, *E*, *R*, and *A* directly.

First consider the number of ways to place the vowels alphabetically in the arrangement.

1. If the first *A* is in the first position, then the second *A* may be in the second, third, fourth, or fifth position. It cannot appear in the sixth position, as the *E* must appear after the second *A*.
  - If the second *A* is in the second position, the *E* can be placed in 4 ways (in the third, fourth, fifth or sixth position).
  - If the second *A* is in the third position, the *E* can be placed in 3 ways (in the fourth, fifth or sixth position).
  - If the second *A* is in the fourth position, the *E* can be placed in 2 ways (in the fifth or sixth position).
  - If the second *A* is in the fifth position, the *E* can be placed in 1 way (in the sixth position).

There is a total of  $4 + 3 + 2 + 1 = 10$  ways to place the vowels in alphabetical order so that the first *A* is in the first position.

2. If the first *A* is in the second position, then the second *A* may be in the third, fourth, or fifth position.
  - If the second *A* is in the third position, the *E* can be placed in 3 ways (in the fourth, fifth or sixth position).
  - If the second *A* is in the fourth position, the *E* can be placed in 2 ways (in the fifth or sixth position).
  - If the second *A* is in the fifth position, the *E* can be placed in 1 way (in the sixth position).

There is a total of  $3 + 2 + 1 = 6$  ways to place the vowels in alphabetical order so the first *A* is in the second position.



3. If the first  $A$  is in the third position, then the second  $A$  may be in the fourth or fifth position.
  - If the second  $A$  is in the fourth position, the  $E$  can be placed in 2 ways (in the fifth or sixth position).
  - If the second  $A$  is in the fifth position, the  $E$  can be placed in 1 way (in the sixth position).There is a total of  $2 + 1 = 3$  ways to place the vowels in alphabetical order so the first  $A$  is in the third position.
4. If the first  $A$  is in the fourth position, then the second  $A$  must be in the fifth position and the  $E$  must be in the sixth position. So there is a total of 1 way to place the vowels in alphabetical order so the first  $A$  is in the fourth position.

Adding each of the above results, there is a total of  $10 + 6 + 3 + 1 = 20$  ways to place the vowels so that they are in alphabetical order. For each of the 20 ways to place the vowels in alphabetical order, we find the number of valid ways to fill the remaining three positions with the consonants so that they are not in alphabetical order.

It turns out that the consonants can be placed in the three remaining positions in six possible orders:  $CMR$ ,  $CRM$ ,  $MCR$ ,  $MRC$ ,  $RCM$ , and  $RMC$ . One of the arrangements of the consonants is in alphabetical order and five of the arrangements are not in alphabetical order. So for each of the 20 arrangements in which the vowels are in alphabetical order there are 5 arrangements of the consonants so they are not in alphabetical order.

Therefore, there are  $20 \times 5 = 100$  arrangements of the letters of the word  $CAMERA$  in which the vowels appear alphabetically and the consonants do not appear alphabetically.

### **Solution 2**

In this solution we will first count the total number of arrangements of the letters in the word  $CAMERA$ .

Notice that there are two  $A$ s. We need to be careful not to count arrangements twice. So we will consider placing the distinct letters  $C$ ,  $E$ ,  $M$ , and  $R$  first. There are 6 places for the  $C$ . For each of these placements of  $C$ , there are 5 ways to place the  $E$ . This gives a total of  $6 \times 5 = 30$  ways to place the  $C$  and  $E$ . For each of these placements of  $C$  and  $E$ , there are 4 ways to place the  $M$ . This gives a total of  $30 \times 4 = 120$  ways to place the  $C$ ,  $E$  and  $M$ . For each of these placements of  $C$ ,  $E$  and  $M$ , there are 3 ways to place the  $R$ . This gives a total of  $120 \times 3 = 360$  ways to place the  $C$ ,  $E$ ,  $M$ , and  $R$ . The two  $A$ s must go in the remaining two spots and this can be done in only 1 way. Therefore, there is a total of 360 ways to arrange the six letters of the word  $CAMERA$ .

In examining the 360 arrangements, you would see the vowels appearing in one of three orders:  $AAE$ ,  $AEA$  and  $EAA$ , only one of which is in alphabetical order. So in one-third of the 360 arrangements, or 120 arrangements, the vowels will appear in alphabetical order.

In examining the 120 arrangements in which the vowels appear alphabetically, the consonants will appear in one of six different orders:  $CMR$ ,  $CRM$ ,  $MCR$ ,  $MRC$ ,  $RCM$ , and  $RMC$ . Five-sixths of these orderings have the consonants out of alphabetical order. So five-sixths of the 120 arrangements, or 100 arrangements, will be such that the vowels appear alphabetically and the consonants do not appear alphabetically.

Therefore, there are 100 arrangements of the letters of the word  $CAMERA$  in which the vowels appear alphabetically and the consonants do not appear alphabetically.