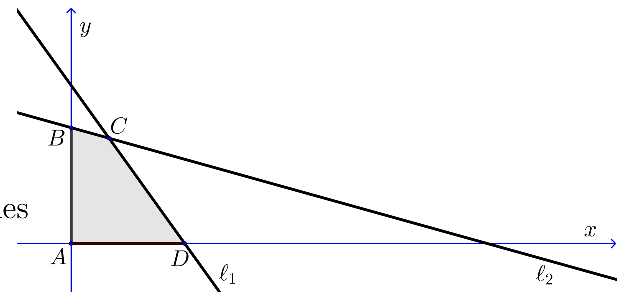
Problem of the Week

Problem D and Solution

The Area Within

Problem

Quadrilateral $ABCD$ is constructed as follows:



- vertex A is located at the origin;
- vertex C is at the intersection point of the lines $\ell_1: \; 5x + 2y = 30$ and $\ell_2: \; x + 2y = 22$;
- vertex B is at the intersection point of the y-axis and ℓ_2; and
- vertex D is at the intersection point of the x-axis and ℓ_1.

Determine the area of $ABCD$.

Solution

Let the coordinates of C be (h, k) where h is the horizontal distance from the y-axis to C and k is the vertical distance from the x-axis to C.

To find the coordinates of D, let $y = 0$ in $5x + 2y = 30$. Therefore, the x-intercept is 6 and the coordinates of D are $(6, 0)$.

To find the coordinates of B, let $x = 0$ in $x + 2y = 22$. Therefore, the y-intercept is 11 and the coordinates of B are $(0, 11)$.

Two methods are provided to find C, the point of intersection of ℓ_1 and ℓ_2.

1. Solving for C using the method of substitution:
   Rewrite equation ℓ_2 as $x = 22 - 2y$.

   Substitute for x in ℓ_1 so that $5(22 - 2y) + 2y = 30$. Simplifying, $110 - 10y + 2y = 30$. This further simplifies to $-8y = -80$ and $y = 10$.

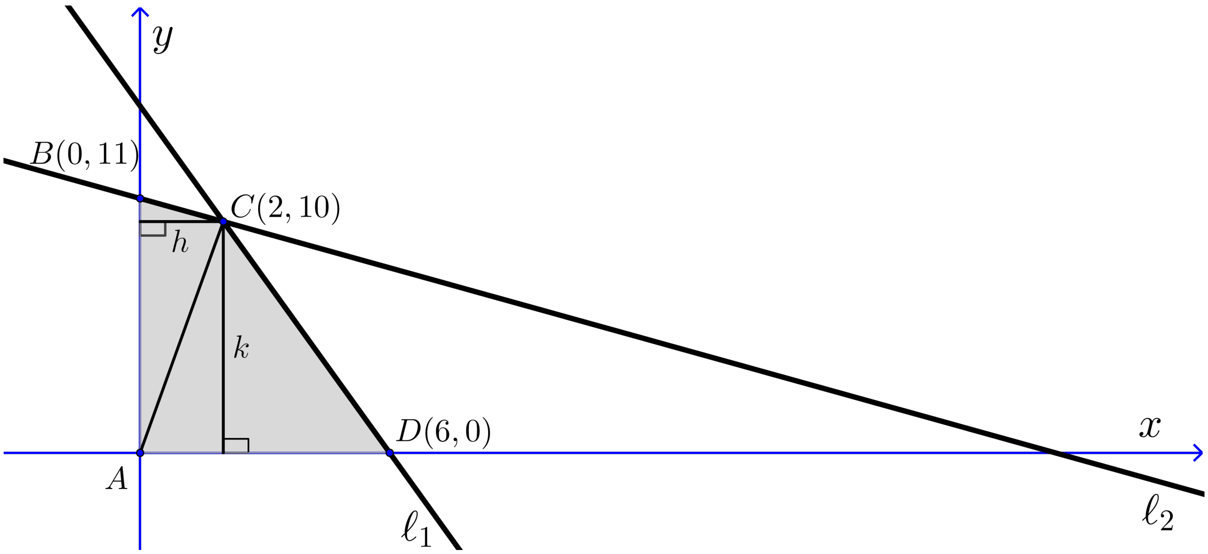
   Substituting $y = 10$ in $x + 2y = 22$ gives $x + 20 = 22$ and $x = 2$.

   The coordinates of C, the point of intersection of ℓ_1 and ℓ_2, are $(2, 10)$. Therefore, $h = 2$ and $k = 10$.

2. Solving for C using the method of elimination:

$$\begin{aligned} \ell_1: \quad & 5x + 2y &= \;& 30 \\ \ell_2: \quad & x + 2y &= \;& 22 \\ \text{Subtracting, we obtain,} \quad & 4x &= \;& 8 \\ \text{Therefore,} \quad & x &= \;& 2 \end{aligned}$$

   Substituting $x = 2$ in l_1, $10 + 2y = 30$ and $y = 10$. The coordinates of C, the point of intersection of ℓ_1 and ℓ_2, are $(2, 10)$. Therefore, $h = 2$ and $k = 10$.



Quadrilateral $ABCD$ can be divided into two triangles, $\triangle ABC$ and $\triangle ACD$. Therefore,

$$\begin{aligned}
\text{Area } ABCD &= \text{Area } \triangle ABC + \text{Area } \triangle ACD \\
&= \frac{1}{2}(h \times AB) \quad + \quad \frac{1}{2}(k \times AD) \\
&= \frac{1}{2}(2)(11) \; + \; \frac{1}{2}(10)(6) \\
&= 11 + 30 \\
&= 41
\end{aligned}$$

Therefore, the area of $ABCD$ is 41 units^2.