Problem of the Week
Problem D and Solution
Stacks and Stacks

Problem

Virat has a large collection of \$2 bills and \$5 bills. He makes stacks that have a value of \$100. Each stack has a least one \$2 bill, at least one \$5 bill, and no other types of bills. If each stack has a different number of \$2 bills than any other stack, what is the maximum number of stacks that Virat can create?

Solution

Consider a stack of bills with a total value of \$100 that includes x \$2 bills and y \$5 bills. The \$2 bills are worth $\$2x$ and the \$5 bills are worth $\$5y$, and so $2x + 5y = 100$.

Determining the number of possible stacks that the teller could have is equivalent to determining the numbers of pairs (x, y) of integers with $x \geq 1$ and $y \geq 1$ and $2x + 5y = 100$ or $5y = 100 - 2x$. (We must have $x \geq 1$ and $y \geq 1$ because each stack includes at least one \$2 bill and at least one \$5 bill.)

Since $x \geq 1$, then:

$$\begin{aligned} 2x &\geq 2 \\ 2x + 98 &\geq 100 \\ 98 &\geq 100 - 2x \end{aligned}$$

This can be rewritten as $100 - 2x \leq 98$.

Also, since $5y = 100 - 2x$, this becomes $5y \leq 98$.

This means that $y \leq \frac{98}{5} = 19.6$. Since y is an integer, then $y \leq 19$.

Notice that since $5y = 100 - 2x$, then the right side is the difference between two even integers and is therefore even. This means that $5y$ (the left side) is itself even, which means that y must be even.

Since y is even, $y \geq 1$, and $y \leq 19$, then the possible values of y are 2, 4, 6, 8, 10, 12, 14, 16, 18.

Each of these values gives a pair (x, y) that satisfies the equation $2x + 5y = 100$. These ordered pairs are:

$(x, y) = (45, 2), (40, 4), (35, 6), (30, 8), (25, 10), (20, 12), (15, 14), (10, 16), (5, 18)$

Therefore, we see that the maximum number of stacks that Virat could have is 9.